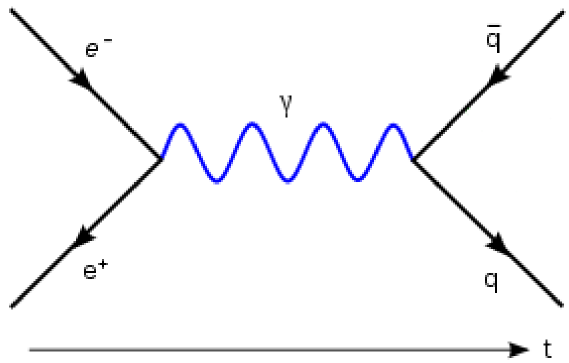

Quantum electrodynamics

Notes of the lecture from
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Contents

Motivation

The cycle in Regensburg contains Quantum Electrodynamics (QED), Quantum Chromodynamics (QCD) and Quantum Field Theory (QFT) with QED as introduction. The basic problem, which is discussed in this cycle, is about the merge of classical Quantum Mechanics, which brings us to Heisenberg's uncertainty principle

$$\Delta E \Delta t \geq \frac{\hbar}{2},$$

with the Special theory of relativity (STR), which includes the energy-momentum relation

$$E^2 = m^2 c^4 + \vec{p}^2 c^2.$$

The TR allows solutions with negativ energies, which brings us to particles / antiparticles. Furthermore the vacuum is becoming a medium, because interactions between particles and their antiparticles are taking place in the vacuum.

The difference to solid state physics / plasma physics is:

- There is mostly a cut-off at small and big impetuses. $\Rightarrow$ finite results.
- QED $E \to 0$, $E \to \infty$ accounts in general. $\Rightarrow$ Infinity.

Recommended literature for this lecture:

- Bjorken, Drell
- Nachtmann
- Peslein, Schröder: Quantum Field Theory
- Greiner, Reinhart: Quantum Electrodynamics (H.Deutsch, 1995)

0 Special relativity

The central axiom of the STR is given through

$$(ds)^2 \equiv c^2(dt)^2 - (d\vec{x})^2 = \text{const.} \tag{0.1}$$

Thus we obtain through some elementary transformation that

$$(ds)^2 \geq 0 \;\Leftrightarrow\; c^2 \geq \left(\frac{d\vec{x}}{dt}\right)^2,$$

which is why the speed of light is the critical velocity.

Light fulfills $(ds)^2 = 0$ in all inertial frames. Let us assume that $x^\mu = (ct, x^1, x^2, x^3)$ and $(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu$, so we get

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We call $(g_{\mu\nu})$ the metric tensor, Minkowski metric or just metric of the special theory of relativity.

By using the Einstein notation we can see that

$$a_\mu = g_{\mu\nu} a^\nu,$$

where a_μ is called a <u>covariant</u> vector and a^ν is called a <u>contravariant</u> vector. With the help of equation (0.1) it is possible to make a transformation from x to x' between to inertial frames I and I'. The result is that

$$(ds)^2 = g_{\alpha\alpha'}dx^\alpha dx^{\alpha'} \stackrel{!}{=} g_{\beta\beta'}(dx')^\beta (dx')^{\beta'}.$$

What we are going to see later is that only linear transformations can fulfull equation (0.1). We will now make an ansatz using the so called Poincare-Group,

$$\begin{aligned}(x')^\beta &= \Lambda^\beta{}_\alpha x^\alpha + a^\beta,\\ (dx')^\beta &= \Lambda^\beta{}_\alpha dx^\alpha.\end{aligned}$$

Inserting this in equation (0.1) brings us to

$$(ds)^2 = g_{\alpha\alpha'}dx^\alpha dx^{\alpha'} = g_{\beta\beta'}\Lambda^\beta{}_\alpha \Lambda^{\beta'}{}_{\alpha'} dx^\alpha dx^{\alpha'}.$$

Thus we can see directly that equation (0.1) is equivalent to

$$\Lambda^\beta{}_\alpha \Lambda^{\beta'}{}_{\alpha'} g_{\beta\beta'} = g_{\alpha\alpha'}. \tag{0.2}$$

Remark Every 'scalar product' $g_{\mu\nu}a^\mu b^\nu \equiv a \cdot b$ is invariant under Lorentz-transformations. We also see that

$$a_\nu b^\nu = a^\mu b_\mu = a \cdot b,$$

with b^ν being defined as four objects, which transform like dx^ν mit $\Lambda^\mu{}_\nu$.

Definition of the vector $\vec{v}$. Let's assume

$$\frac{v^i}{c} \equiv \frac{d(x')^i}{c(dt')} \overset{=}{dx^i} = 0 \; \frac{\Lambda^i{}_0 c dt}{\Lambda^0{}_0 c dt} = \frac{\Lambda^i{}_0}{\Lambda^0{}_0}.$$

Insertation of equation (0.2) with $\alpha = \alpha' = 0$ results in

$$\begin{aligned}\Lambda^\beta{}_0 \Lambda^{\beta'}{}_0 g_{\beta\beta'} &= 1,\\ (\Lambda^0{}_0)^2 - (\Lambda^j{}_0)^2 &= 1,\\ \Rightarrow (\Lambda^0{}_0)^2 \left[1 - \left(\frac{v^i}{c}\right)^2\right] &= 1,\\ \Rightarrow \Lambda^0{}_0 &= \pm\frac{1}{\sqrt{1-\beta^2}} \equiv \pm\gamma, \qquad \beta^2 \equiv \frac{\vec{v}^2}{c^2}.\end{aligned}$$

The minus sign represents a time reversal including normal Lorentz-transformations. Initially we will only discuss the '+'-sign, i.e. no time reversal. For the other time components we get

$$\Lambda^i{}_0 = \frac{v^i}{c\sqrt{1-\frac{\vec{v}}{c^2}}} = \gamma\beta^i.$$

We will now discuss why we want to focus on the '+'-sign. From equation (0.2) we get

$$(\det\Lambda)^2 = 1 \quad \Rightarrow \quad \det\Lambda = \pm 1.$$

For $\alpha = \alpha' = 0$ we calculcate that

$$\Lambda^\beta{}_0\Lambda^{\beta'}{}_0 g_{\beta\beta'} = 1 = g_{00} \quad \text{and} \quad \Lambda^0{}_0\Lambda^0{}_0 - \Lambda^j{}_0\Lambda^j{}_0 = 1, \quad j = 1,2,3.$$

We see directly that

$$(\Lambda^0{}_0)^2 = 1 + (\Lambda^j{}_0)^2 \geq 1 \qquad \Rightarrow \Lambda^0{}_0 \geq 1, \quad \text{or } \Lambda^0{}_0 \leq -1.$$

Next to the Lorentz-transformations with $\det\Lambda = +1, \Lambda^0{}_0 \geq 0$ are the discret Lorentz-transformations, like the Parity (point reflection)

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

or the Time reversal

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can illustrate the Lorentz-Group in the following four sections:

- $L_+^\uparrow$: $\det\Lambda = 1$ and $\Lambda^0{}_0 \geq 1$.

- $L_-^\uparrow$: $\det\Lambda = -1$ and $\Lambda^0{}_0 \geq 1$.
- $L_+^\downarrow$: $\det\Lambda = 1$ and $\Lambda^0{}_0 \leq -1$.
- $L_-^\downarrow$: $\det\Lambda = -1$ and $\Lambda^0{}_0 \leq -1$.

We see that $L_+^\uparrow$ is connected to $L_-^\uparrow$ with $\mathcal{P}$, whereas $L_+^\uparrow$ is connected to $L_-^\downarrow$ with $\mathcal{T}$ and $L_+^\uparrow$ is connected to $L_-^\downarrow$ with $\mathcal{PT}$.

We now show that $\Lambda^i{}_0 = \Lambda^0{}_i$. We can do that again by using equation (0.2) with $\alpha = 0$ and $\alpha' = i$. We get

$$0 = \Lambda^0{}_0\Lambda^0{}_i - \Lambda^j{}_0\Lambda^j{}_i \quad \Rightarrow \quad 0 = \Lambda_{00}\Lambda_{0i} - \Lambda_{j0}\Lambda_{ji}.$$

Now we use equation (0.2) with $\alpha = i, \alpha' = j$ and obtain

$$\delta_{ij} = -\Lambda_{0i}\Lambda_{0j} + \Lambda_{li}\Lambda_{lj}.$$

By multiplying this with $\sum_{i,j}\Lambda_{i0}\Lambda_{j0}$ and using $\Lambda_{ji} = \Lambda_{ij}$ (proof later) we finally get

$$\Lambda_{i0}\Lambda_{i0} = -(\Lambda_{i0}\Lambda 0i)^2 + (\Lambda_{00})^2\Lambda_{0l}\Lambda_{0l}.$$

We already know that the $\Lambda^\mu{}_\nu$ only depend on the vector $\vec{v}$. On the other side 'i' in $\Lambda^i{}_0$ is one three-vector-component.

<u>Ansatz</u> We know that $\Lambda^i{}_0 \propto v^i \propto \Lambda_0^i$. So we know that $\Lambda^i{}_0 = \text{const}\Lambda_0{}^i$. Inserting this in the equation above brings us to

$$\Lambda_{i0}\Lambda_{i0} = -\text{const}^2\left(-(\Lambda_{i0}\Lambda_{i0}) + (\Lambda_{00})^2\right) = \text{const}^2 \quad \Rightarrow \text{const} = \pm 1.$$

We have a special case when $v^2 = v^3 = 0$. We see that const $= -1$. This leads us to $\Lambda^i{}_0 = -\Lambda_0{}^i$ which is equivalent to

$$\Lambda^i{}_0 = \Lambda^0{}_i.$$

Thus we found that Λ is symmetric. The space components are more complex due to the possibility of rotation. Therefore we need a rotation matrix, which we can gather through a seperation of velocity and rotation. Such rotation matrices must be anti-symmetric.

Here is the sketch of a proof for the space components. We already know that rotation matrices look like (example: rotation around the z-axis)

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & \sin(\varphi) & 0 \\ 0 & -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the next part we seperate ${\Lambda^i}_j$ in the symmetric and antisymmetric parts for $i \neq j$, so that

$${\Lambda^i}_j = bv^i v_j + c{g^i}_j.$$

After a detailled examination we obtain that

$${\Lambda^k}_i = \delta_{ik} + v_i v_k \frac{\gamma - 1}{\vec{v}^2}.$$

So we can now build a general Lorentz-transformation matrix,

$$({\Lambda^\beta}_\alpha) = \begin{pmatrix} \gamma & \frac{v_1}{c}\gamma & \frac{v_2}{c}\gamma & \frac{v_3}{c}\gamma \\ \frac{v_1}{c}\gamma & 1 + \frac{(v_1)^2}{\vec{v}^2}(\gamma - 1) & \frac{v_1 v_2}{\vec{v}^2}(\gamma - 1) & \frac{v_1 v_3}{\vec{v}^2}(\gamma - 1) \\ \frac{v_2}{c}\gamma & \frac{v_1 v_2}{\vec{v}^2}(\gamma - 1) & 1 + \frac{(v_2)^2}{\vec{v}^2}(\gamma - 1) & \frac{v_2 v_3}{\vec{v}^2}(\gamma - 1) \\ \frac{v_3}{c}\gamma & \frac{v_1 v_3}{\vec{v}^2}(\gamma - 1) & \frac{v_2 v_3}{\vec{v}^2}(\gamma - 1) & 1 + \frac{(v_3)^2}{\vec{v}^2}(\gamma - 1) \end{pmatrix}.$$

Proof that $x_\mu \rightarrow x'_\mu$ is linear. We start with $(ds)^2 = (ds')^2$. Then we calculate

$$\begin{aligned} g_{\beta\beta'}(dx')^\beta(dx')^{\beta'} &= g_{\beta\beta'} \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial x'^{\beta'}}{\partial x^{\alpha'}} dx^\alpha dx^{\alpha'} \\ &\stackrel{!}{=} g_{\alpha\alpha'} dx^\alpha dx^{\alpha'} \\ \Rightarrow g_{\alpha\alpha'} &= g_{\beta\beta'} \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial x'^{\beta'}}{\partial x^{\alpha'}} \quad | \cdot \frac{\partial}{\partial x^\gamma} \\ 0 &= g_{\beta\beta'} \left(\frac{\partial^2 x'^\beta}{\partial x^\gamma \partial x^\alpha} \frac{\partial x'^{\beta'}}{\partial x^{\alpha'}} + \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial^2 x'^{\beta'}}{\partial x^\gamma \partial x^{\alpha'}} \right) \\ 0 &= g_{\beta\beta'} \left(\frac{\partial^2 x'^\beta}{\partial x^{\alpha'} \partial x^\alpha} \frac{\partial x'^{\beta'}}{\partial x^\gamma} + \frac{\partial x'^\beta}{\partial x^\alpha} \frac{\partial^2 x'^{\beta'}}{\partial x^{\alpha'} \partial x^\gamma} \right) \\ \Rightarrow 0 &= 2g_{\beta\beta'} \frac{\partial^2 x'^\beta}{\partial x^\alpha \partial x^\gamma} \frac{\partial x'^\beta}{\partial x^{\alpha'}}. \end{aligned}$$

Thus we see that x' has to be a linear function of x to fulfill the requirements.

Four-vectors have a well-defined transformation behaviour. Requirement All physical relevent quantities must be four-vectors or tensors, e.g.

$$p^\mu = (E, \vec{p}), \qquad (c = \hbar = 1).$$

For $\vec{E}$ and $\vec{B}$ we get six components and the four Maxwell equations,

$$\begin{aligned} \nabla \vec{E} &= 4\pi \varrho, \\ \nabla \times \vec{B} &= \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \\ \nabla \times \vec{E} &= -\frac{1}{c}\dot{\vec{B}}, \\ \nabla \vec{B} &= 0. \end{aligned}$$

We can return these equations to the four-vector potential A^μ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. By doing this we obtain

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix},$$

which we could use with the definition of the potential $\vec{A}$, which is

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla A^0 - \frac{1}{c}\dot{\vec{A}}.$$

So we automatically get $\nabla \vec{B} = \nabla(\nabla \times \vec{A}) = 0$ as well as $\nabla \times \vec{E} = -\frac{1}{c}\frac{d}{dt}(\nabla \times \vec{A}) = -\frac{1}{c}\dot{\vec{B}}$. This makes the Maxwell equations obsolete. The two remaining Maxwell equations can be summarised as

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta, \qquad j^\beta = (\varrho, \vec{j}).$$

1 The Dirac equation

We will now work in natural units $\hbar = c = 1$, where 197.3 MeVfm≈1 as well as $2.99979 \cdots 10^8$ m/s≈1. We will now make a first approach to build a relativistic quantum theory. Therefore we combine the relativistic energy-momentum equation with equations providing plane-waves as solutions.

$$\begin{aligned} E^2 &= \vec{p}^2 + m^2, \\ i\frac{\partial}{\partial t}\exp(-i(Et - \vec{p}\vec{x})) &= E\exp(-i(Et - \vec{p}\vec{x})) \text{ and} \\ i\nabla\exp(-i(Et - \vec{p}\vec{x})) &= \vec{p}\exp(-i(Et - \vec{p}\vec{x})). \end{aligned}$$

The combination of these three equations gives us

$$\left[-\left(\frac{\partial}{\partial t}\right)^2 + \nabla^2 - m^2\right]\Phi(\underbrace{t, \vec{x}}_{x}) = 0,$$

which is the Klein-Gordon equation describing Spin-0 particles. Another ansatz would have been to use

$$E = \sqrt{\vec{p}^2 + m^2} \longrightarrow \sqrt{-\nabla^2 + m^2}.$$

The problem is that this method requires a series expansion, where we get derivatives of any order. The Taylor series would bring us to

$$f(x + y) = f(x) + f'(x)y + +f''(x)\frac{y^2}{2!} + ...,$$

which has no strong localization. This is in conflict to causality! Through linearisation we get the Dirac equation. Consider $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ with $\gamma^3 = -\gamma_3, \gamma^2 = -\gamma_2, \gamma^1 = -\gamma_1, \gamma^0 = \gamma_0$ objects that fulfill

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}.$$

Multiplication with $g^{\mu' \mu}$ gives us

$$\gamma^{\mu'} \gamma_\nu + \gamma_\nu \gamma^{\mu'} = 2 g^{\mu'}_{\ \nu}.$$

Then we get

$$\begin{aligned}
(\hat{p}^\mu \gamma_\mu - m)(\hat{p}^\nu \gamma_\nu + m) &= (\hat{p}^\nu \gamma_n u + m)(\hat{p}^\mu \gamma_\mu - m) = \\
&= \hat{p}^\mu \hat{p}^\nu \gamma_\mu \gamma_\nu - m^2 = \frac{1}{2}(\hat{p}^\mu \hat{p}^\nu + \hat{p}^\nu \hat{p}^\mu)\gamma_\mu \gamma_\nu - m^2 = \\
&= \frac{1}{2}\hat{p}^\mu \hat{p}^\nu (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) - m^2 = \hat{p}^2 - m^2.
\end{aligned}$$

If $(\not{p} \pm m)\psi = 0$ is true, with the definition that $a^\mu \gamma_\mu \equiv \not{a}$, then

$$(\not{p} \mp m)(\not{p} \pm m)\psi = (p^2 - m^2)\psi = 0$$

is true as well, which is the Klein-Gordon equation. Furthermore if Φ is a solution of the Klein-Gordon equation we find that

$$\psi \equiv (\hat{\not{p}} - m)\Phi(x)$$

is also a solution of $(\hat{\not{p}} + m)\psi(x) = 0$ and

$$\psi' \equiv (\hat{\not{p}} + m)\Phi(x)$$

is also a solution of $(\hat{\not{p}} - m)\psi(x) = 0$.

We will use the Dirac representation of the γ matrices which are 4×4 matrices, with

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = -\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\gamma^2 = -\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = -\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Our ψ has four components and is called a *Spinor*. We now view $\vec{p} = \vec{0}$ in the rest frame, with a plane wave

$$\psi_1(x^0, \vec{x}) = u(\vec{p} = 0) \exp(-iEt + 0).$$

By inserting this we obtain

$$\begin{aligned} 0 &\stackrel{!}{=} (\hat{\not{p}} - m)\psi_1(x) = (E\gamma^0 - m)\psi_1 = m \begin{pmatrix} 1-1 & 0 & 0 & 0 \\ 0 & 1-1 & 0 & 0 \\ 0 & 0 & -1-1 & 0 \\ 0 & 0 & 0 & -1-1 \end{pmatrix} \psi_1 = \\ &= m \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \psi_1. \end{aligned}$$

So we have two independent solutions

$$u(\vec{p} = 0, +) = \text{const} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u(\vec{p} = 0, -) = \text{const} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The two upper components are equivalent to the Pauli wave functions in the non-relativistic limit case. Else we get

$$(\hat{\not{p}} + m)\psi = 0,$$

where the first and second component is zero. We will see that this is equivalent to the Pauli equation for the positron. It is also possible to obtain the Pauli equation by replacing E with E. For the other case we get

$$\psi_2(x^0, \vec{x}) = v \exp(iEt + 0) \quad \Rightarrow \quad 0 \stackrel{!}{=} (\hat{\not{p}} - m)\psi_2(x) = -(E\gamma^0 + m)\psi_2.$$

We see that negative energy solutions have something to do with antiparticles. We will have a closer look at that later - *Charge Reversal*.

1.1 Lorentz transformation of the Dirac equation

Demands The dirac equation should be the same in every inertial frame. So we demand that

$$0 = \left(i\frac{\partial}{\partial x_\mu}\gamma_\mu\right)\psi(x) = \left(i\frac{\partial}{\partial x'_\nu}\Lambda_\nu{}^\mu\gamma_\mu - m\right)\underbrace{S^{-1}(\Lambda)\psi'(x')}_{\psi(x)}, \tag{1.1}$$

where $S^{-1}(\Lambda)$ is a arbitrary 4×4 matrix.

Remark We see that $\frac{\partial}{\partial x_\mu} = \partial^\mu$, e.g.

$$\frac{\partial}{\partial x_\mu}(x \cdot y) = \frac{\partial}{\partial x_\mu}(x_\nu y^\nu) = y^\mu,$$

and

$$\begin{aligned} \Lambda^\alpha{}_{\alpha'}\Lambda^\beta{}_{\beta'}g_{\alpha\beta} &= g_{\alpha'\beta'}, \\ \Rightarrow \Lambda^\alpha{}_{\alpha'}\Lambda^{\beta\beta'}g_{\alpha\beta} &= g_{\alpha'}^{\ \beta'} = \delta^{\beta'}_{\alpha'}, \\ \Rightarrow \Lambda^\alpha{}_{\alpha'}\Lambda_\alpha{}^{\beta'} &= \delta^{\beta'}\alpha'. \end{aligned}$$

We still have to show that from equation (1.1) we get to

$$\left(i\frac{\partial}{\partial x'_\nu}\gamma_\nu - m\right)\psi'(x') = 0.$$

By multiplying S with equation (1.1) we obtain

$$0 = (i\frac{\partial}{\partial x'_\nu}\underbrace{\Lambda_\nu{}^\mu S(\Lambda)\gamma_\mu S^{-1}(\Lambda)}_{\stackrel{!}{=}\gamma_\nu} - m)\psi'(x').$$

To begin with we will look at infinitesimal transformations,

$$\Lambda^{\mu\nu}_{\text{infinitesimal}} = g^{\mu\nu} + \frac{\omega^{\mu\nu}}{N} + \mathcal{O}\left(\frac{1}{N^2}\right).$$

We use that

$$\lim_{N\to\infty}\left[1 - i\frac{a}{N}\right]^N = \exp(-ia) \quad \text{and} \quad \Lambda^{\mu}{}_{\mu'} g_{\mu\nu} \Lambda^{\nu}{}_{\nu'} = g_{\mu'\nu'}.$$

Thus we find that

$$0 = g^{\mu}{}_{\mu'} g_{\mu\nu} \frac{\omega^{\nu}_{\nu'}}{N} + \frac{\omega^{\mu}{}_{\mu'}}{N} g_{\mu\nu} g^{\nu}{}_{\nu'}.$$

This brings us to

$$\frac{\omega_{\mu'\nu'}}{N} + \frac{\omega_{\nu'\mu'}}{N} = 0,$$

what gives us the condition that ω has to be antisymmetric, i.e.

$$\omega_{\mu'\nu'} = -\omega_{\nu'\mu'}.$$

So we make the ansatz for

$$S = 1 - \frac{i}{4}\frac{\omega^{\mu\nu}}{N}\sigma_{\mu\nu} + \mathcal{O}\left(\frac{1}{N^2}\right),$$

where the $\sigma_{\mu\nu}$ are 16 4×4 matrices. We pick only the six antisymmetric ones, where $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$. Our demand from above now looks like

$$\left(1 - \frac{i}{4}\sigma_{\mu'\nu'}\frac{\omega^{\mu'\nu'}}{N}\right)\left(\gamma_\nu + \frac{\omega_{\nu\mu}}{N}\gamma^\mu\right)\left(1 + \frac{i}{4}\sigma_{\mu''\nu''}\frac{\omega^{\mu''\nu''}}{N}\right) \stackrel{!}{=} \gamma_\nu + \mathcal{O}\left(\frac{1}{N^2}\right).$$

Consequently we obtain

$$0 = -\frac{i}{4}\sigma_{\mu'\nu'}\frac{\omega^{\mu'\nu'}}{N}\gamma_\nu + \frac{\omega_{\nu}{}^{\mu}}{N}\gamma_\mu + \frac{i}{4}\gamma_\nu\sigma_{\mu''\nu''}\frac{\omega^{\mu''\nu''}}{N}.$$

After a simple algebraic manipulation and renaming of some indices we finally get

$$\frac{\omega^{\mu'\nu'}}{N}\left(-\frac{i}{4}\sigma_{\mu'\nu'}\gamma_\nu - g_{\nu\nu'}\gamma_{\mu'} + \frac{i}{4}\gamma_\nu\sigma_{\mu'\nu'}\right) = 0.$$

By using that

$$-g_{\nu\nu'}\gamma_{\mu'} = -\frac{1}{2}g_{\nu\nu'}\gamma_{\mu'} + \frac{1}{2}g_{\nu\mu'}\gamma_{\nu'},$$

we only have to show that there are $\sigma_{\mu\nu}$ matrices which fulfill the equation

$$\frac{i}{2}(\sigma_{\mu'\nu'}\gamma_{\nu} - \gamma_{\nu}\sigma_{\mu'\nu'}) = g_{\mu'\nu}\gamma_{\nu'} - g_{\nu'\nu}\gamma_{\mu'}.$$

There are only two possibilities:

- The first possibility is that

$$\sigma_{\mu'\nu'} = \text{const}(\gamma_{\mu'}\gamma_{\nu'} - \gamma_{\nu'}\gamma_{\mu'}).$$

- The second possibility is that it is the first possibility multiplied with γ_5.

We will now show that the first possibility is correct. So we insert this one in the left hand side of our equation and get

$$\begin{aligned}
\text{lhs} &= \frac{i}{2}\text{const}\left(\gamma_{\mu'}\gamma_{\nu'}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu'}\gamma_{\nu'} - \gamma_{\nu'}\gamma_{\mu'}\gamma_{\nu} + \gamma_{\nu}\gamma_{\nu'}\gamma_{\mu'}\right) = \\
&= \frac{i}{2}\text{const}(2g_{\nu\nu'}\gamma_{\mu'} - \gamma_{\mu'}\gamma_{\nu}\gamma_{\nu'} - 2g_{\nu\mu'}\gamma_{\nu'} + \gamma_{\mu'}\gamma_{\nu}\gamma_{\nu'} - \\
&- 2g_{\mu'\nu}\gamma_{\nu'} + \gamma_{\nu'}\gamma_{\nu}\gamma_{\mu'} + 2g_{\nu\nu'}\gamma_{\mu'} - \gamma_{\nu'}\gamma_{\nu}\gamma_{\mu'}) = \\
&= 2i\text{const}\left(g_{\nu\nu'}\gamma_{\mu'} - g_{\nu\mu'}\gamma_{\nu'}\right), \\
\Rightarrow \text{const} &= \frac{i}{2} \quad \Rightarrow \quad \sigma_{\mu'\nu'} = \frac{i}{2}[\gamma_{\mu'}, \gamma_{\nu'}].
\end{aligned}$$

1.1.1 Rotation

We will now use our knowledge to calculate the Lorentz-Transformation of three-dimensional rotations.

Reminder From the classic electrodynamic we already know that

$$Y_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l{}^m(\cos\vartheta)\exp(im\varphi).$$

We also know from quantum mechanics that

$$\hat{L}_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial \varphi}.$$

The transformation matrix for a rotation around the z-axis looks like

$$(\Lambda^{\mu}{}_{\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{\varphi}{N} & 0 \\ 0 & \frac{\varphi}{N} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can see that

$$\frac{\omega^{12}}{N} = -\frac{\omega^{21}}{N} = -\frac{\varphi}{N}.$$

All other $\omega_{\mu\nu}$ are zero. So we only need

$$\begin{aligned} \sigma_{12} &= -\sigma_{21} = \frac{i}{2}(\gamma_1\gamma_2 - \gamma_2\gamma_1) = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Now we can construct the finite transformation, which is

$$S = \exp\left(-\frac{i}{2}N\frac{\omega^{12}}{N}\sigma_{12}\right) = \exp\left(\frac{i}{2}\varphi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\right).$$

For further investigation we need to know how σ_{12} develops while multiplying it with itself. We see that for an even power factor we get $\sigma_{12} = 1$ and for an odd factor we get $\sigma_{12} = \sigma_{12}$. This brings us to

$$S = 1\cos\frac{\varphi}{2} + i\sigma_{12}\sin\frac{\varphi}{2} = \begin{pmatrix} \exp(i\frac{\varphi}{2}) & 0 & 0 & 0 \\ 0 & \exp(-i\frac{\varphi}{2}) & 0 & 0 \\ 0 & 0 & \exp(i\frac{\varphi}{2}) & 0 \\ 0 & 0 & 0 & \exp(-i\frac{\varphi}{2}) \end{pmatrix}.$$

So we naturally get the Spin of $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$.

1.1.2 Boost

We now look at a boost without rotation. We already know that

$$\begin{aligned} (x^0)' &= \gamma(x^0 + \vec{\beta}\vec{x}), \\ (\vec{x})' &= \gamma(\vec{x} + \vec{\beta}x^0), \qquad \vec{\beta} = \frac{\vec{v}}{c}. \end{aligned}$$

We now define the rapidity ω in order to get $\gamma = \cosh\omega$ and $\beta\gamma = \sinh\omega$. So we can add two boosts which go into the same direction directly and do not have to multiply the matrices. The big advantage is that we can use the same formalism as before. So we get

$$\frac{\omega^0{}_1}{N} = \frac{\omega^1{}_0}{N} = \frac{\omega}{N}.$$

We already know that the transformation matrix looks like

$$(\Lambda^\beta{}_\alpha) = \begin{pmatrix} \gamma & \frac{v_1}{c}\gamma & \frac{v_2}{c}\gamma & \frac{v_3}{c}\gamma \\ \frac{v_1}{c}\gamma & 1 + \frac{(v_1)^2}{\vec{v}^2}(\gamma - 1) & \frac{v_1 v_2}{\vec{v}^2}(\gamma - 1) & \frac{v_1 v_3}{\vec{v}^2}(\gamma - 1) \\ \frac{v_2}{c}\gamma & \frac{v_1 v_2}{\vec{v}^2}(\gamma - 1) & 1 + \frac{(v_2)^2}{\vec{v}^2}(\gamma - 1) & \frac{v_2 v_3}{\vec{v}^2}(\gamma - 1) \\ \frac{v_3}{c}\gamma & \frac{v_1 v_3}{\vec{v}^2}(\gamma - 1) & \frac{v_2 v_3}{\vec{v}^2}(\gamma - 1) & 1 + \frac{(v_3)^2}{\vec{v}^2}(\gamma - 1) \end{pmatrix}.$$

Now we take a closer look at the infinitesimal transformation matrix

$$(\Lambda^\beta{}_\alpha)_{\text{infinitesimal}} = \begin{pmatrix} 1 & \omega^1 & \omega^2 & \omega^3 \\ \omega^1 & 1 & 0 & 0 \\ \omega^2 & 0 & 1 & 0 \\ \omega^3 & 0 & 0 & 1 \end{pmatrix},$$

with $\omega^j = \frac{p^j}{|\vec{p}|}\omega$. We now perform the transition from the infinitesimal to the finite transformation,

$$S = \lim_{N\to\infty}\left[1 - \frac{i}{4}\sigma_{\mu\nu}\frac{\omega^{\mu\nu}}{N}\right]^N = \exp\left(-\frac{i}{2}\sigma_{j0}\frac{p^j}{|\vec{p}|}\omega\right).$$

With $j = 1$ the get

$$\sigma_{10} = \frac{i}{2}\gamma_1\gamma_0 2 = i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

By doing the same for $j = 2$ we obtain

$$\sigma_{20} = \frac{i}{2}\gamma_2\gamma_0 2 = i\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

And finally for $j = 3$ we receive

$$\sigma_{30} = \frac{i}{2}\gamma_3\gamma_0 2 = i\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

So totally we found

$$S = \lim_{N\to\infty}\left[1 + \frac{1}{2}\frac{\omega}{N}\underbrace{\begin{pmatrix} 0 & 0 & p^3 & p^1 - ip^2 \\ 0 & 0 & p^1 + ip^2 & -p^3 \\ p^3 & p^1 - ip^2 & 0 & 0 \\ p^1 + ip^2 & -p^3 & 0 & 0 \end{pmatrix}}_{\equiv M}\right]^N = \exp\left(\frac{\omega}{2}M\right).$$

By making a series expansion we find that

$$M^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M^{2n} = 1, \qquad M^{2n+1} = M.$$

That means that

$$S = 1 \cosh\left(\frac{\omega}{2}\right) + M \sinh\left(\frac{\omega}{2}\right).$$

In order to simplify our result we have to view a finite boost transformation matrix. We already know that

$$(\Lambda^{\beta}{}_{\alpha}) = \lim_{N\to\infty} \left[1 + \frac{\omega}{N} \underbrace{\begin{pmatrix} 0 & \omega^1 & \omega^2 & \omega^3 \\ \omega^1 & 0 & 0 & 0 \\ \omega^2 & 0 & 0 & 0 \\ \omega^3 & 0 & 0 & 0 \end{pmatrix}}_{\equiv \tilde{M}} \right]^N = \exp(\omega \tilde{M}).$$

By comparing M with $\tilde{M}$ we see that

$$\tilde{M}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(\omega^1)^2}{\omega^2} & 0 & 0 \\ 0 & 0 & \frac{(\omega^2)^2}{\omega^2} & 0 \\ 0 & 0 & 0 & \frac{(\omega^3)^2}{\omega^2} \end{pmatrix} \quad \Rightarrow \quad [\tilde{M}^2]^0{}_0 = 1 \Rightarrow [\tilde{M}^{2n}]^0{}_0 = 1,$$

and furthermore $[\tilde{M}^{2n+1}]^0{}_0 = 0$. This together with $\Lambda^0{}_0 = \cosh\omega \stackrel{!}{=} \gamma$ and the relations

$$\begin{aligned} \cosh\frac{\omega}{2} &= \sqrt{\frac{\cosh\omega + 1}{2}} = \sqrt{\frac{\gamma+1}{2}} = \sqrt{\frac{E+m}{2m}}, \\ \frac{1}{|\vec{p}|}\sinh\frac{\omega}{2} &= \frac{1}{|\vec{p}|}\sqrt{\frac{\cosh\omega - 1}{2}} = \frac{1}{\sqrt{(E-m)(E+m)}}\sqrt{\frac{E-m}{2m}} = \sqrt{\frac{E+m}{2m}}\frac{1}{E+m}, \end{aligned}$$

bring us to

$$S = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & 0 & \frac{p_3}{E+m} & \frac{p_1-ip_2}{E+m} \\ 0 & 1 & \frac{p_1+ip_2}{E+m} & \frac{-p_3}{E+m} \\ \frac{p_3}{E+m} & \frac{p_1-ip_2}{E+m} & 1 & 0 \\ \frac{p_1+ip_2}{E+m} & \frac{-p_3}{E+m} & 0 & 1 \end{pmatrix}.$$

Thus we finally know all plane wave solutions:

$$\exp(-ip\cdot x)u(p,+) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_1+ip_2}{E+m} \end{pmatrix} \exp(-ip\cdot x), \quad \exp(-ip\cdot x)u(p,-) = ...,$$

and for the antiparticles

$$\exp(ip\cdot x)v(p,-) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \frac{p_3}{E+m} \\ \frac{p_1+ip_2}{E+m} \\ 1 \\ 0 \end{pmatrix} \exp(ip\cdot x), \quad \exp(ip\cdot x)v(p,+) = ...$$

1.1.3 The spin fourvector

We will now investigate the question what spin is in four dimensions $\rightarrow s^\mu = (?, ...)$. We already know that in the rest frame $s^\mu = (0, \vec{S})$. We are going to use the lorentz-invariance. Outgoing from the constraints

$$\begin{aligned} s\cdot p &= 0, \qquad \text{where } p^\mu = (E, \vec{0}), \\ s\cdot s &= -1, \qquad \text{if } |\vec{s}| = 1. \end{aligned}$$

By using the ansatz that $s^\mu = (s^0, \alpha\vec{p})$ we only have two unknown variables. These two variables can be found using the two constraints. Lorentz invariant means that all scalars $a_\mu b^\mu$, π, $\sigma_{\mu\nu}\omega^{\mu\nu}$ are invariant under Lorentz transformation. From

$$S^0 E - \vec{p}\vec{p}\alpha = 0 \quad \Rightarrow \quad S^0 = \frac{\vec{p}\vec{p}}{E}\alpha$$

we get that

$$(S^\mu) = \alpha \left(\frac{\vec{p}\vec{p}}{E}, \vec{p} \right).$$

From the second relation we can obtain - using that in the rest frame $s_\mu s^\mu = -1$, if $\vec{s}^2 = 1$ - that

$$\begin{aligned} -1 &\stackrel{!}{=} \alpha^2 \left(\frac{(|\vec{p}|^2)^2}{E^2} - \frac{|\vec{p}|^2 E^2}{E^2} \right) = \alpha^2 \frac{|\vec{p}|^2 (|\vec{p}|^2 - E^2)}{E^2} = \\ &= -\alpha^2 \frac{m^2 |\vec{p}|^2}{E^2} \qquad \Rightarrow \alpha = \pm \frac{E}{m|\vec{p}|}. \end{aligned}$$

So finally we found the spin fourvector, which is

$$(S^\mu) = \pm \frac{E}{m} \left(\frac{|\vec{p}|}{E}, \frac{\vec{p}}{|\vec{p}|} \right). \tag{1.2}$$

A remarkable property of this fourvector is that

$$\lim_{\frac{m}{|\vec{p}|} \to 0} (S^\mu) = \pm \frac{1}{m} (E, \vec{p}) = \pm \frac{1}{m} (p^\mu).$$

1.2 Projection operators

The main idea behind constructing projection operators is that

$$\vec{v} = \sum_i \vec{e}_i \left(\vec{e}_i^T \vec{v} \right) = \sum_i \vec{e}_i v_i = \sum_i P_i \vec{v},$$

where P_i is the projection operator on the ith unit vector. The main task is now the transition to the function room of the solutions of the Dirac equation. The first step is to know what is the scalar product in this formalism. Our ansatz is

$$\int d^3x \psi^{*T}(t, \vec{x}) \Gamma \psi(t, \vec{x}) = 1.$$

In this ansatz we have a arbitrary Γ, which represents a 4×4 matrix, and the normalization factor 1. Since our constraction should be lorentz invarianet, we demand that

$$S^\dagger \Gamma S \stackrel{!}{=} \Gamma.$$

In order to Γ to fulfill this relation we need to know $S^\dagger$, which is

$$-\frac{\omega^{\mu\nu}}{4}\left(i\sigma_{\mu\nu}\right)^\dagger = \frac{\omega^{\mu\nu}}{8}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right)^\dagger = \frac{\omega^{\mu\nu}}{8}\left(\gamma_\nu^\dagger\gamma_\mu^\dagger - \gamma_\mu^\dagger\gamma_\nu^\dagger\right).$$

We now use $\gamma_0 = \gamma_0^\dagger$ and $\gamma_i = -\gamma_i^\dagger$ in order to get to

$$\gamma_\nu^\dagger = \gamma_0\gamma_\nu\gamma_0. \tag{1.3}$$

So finally we know that

$$S^\dagger = -\frac{\omega^{\mu\nu}}{4}\left(i\sigma_{\mu\nu}\right)^\dagger = \gamma_0\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\gamma_0.$$

Therefore we know the necessary Γ, which is

$$S^\dagger\Gamma S = \gamma_0 S^{-1}\gamma_0\Gamma S \stackrel{!}{=} \Gamma \quad \Rightarrow \quad \Gamma = \gamma_0.$$

Now we know the scalar product of the Dirac theory

$$\int_V d^3x\, \underbrace{\psi^\dagger(t,\vec{x})\gamma_0}_{\equiv\overline{\psi}(t,\vec{x})}\psi(t,\vec{x}). \tag{1.4}$$

First we will construct the projection operator in the rest frame where $(p^\mu) = (m, \vec{0})$. We already know that

$$P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\left[(1\ 0\ 0\ 0)\gamma_0\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We will now try to construct operators which will fulfill the needed projection properties. From observations we know that the following expression is lorentz invariant

$$\frac{p\cdot\gamma+m}{2m}=\frac{m}{2m}\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=p_1+p_2.$$

Now we choose the $\vec{e}_3$ axis as quantification axis, which gives us

$$\begin{aligned}\gamma_\mu S^\mu &= \gamma_3=\begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix},\gamma_5=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\\ \Rightarrow\gamma_5\gamma_\mu S^\mu &= \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}.\end{aligned}$$

So we found the following lorentz invariant expression,

$$\frac{1+\gamma_5\not{s}}{2}=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},$$

which we can use in combination with the first expression in order to get the projection operators,

$$\begin{aligned}\hat{P}_1 &= \frac{\not{p}+m}{2m}\frac{1+\gamma_5\not{s}}{2},\quad \hat{P}_2=\frac{\not{p}+m}{2m}\frac{1-\gamma_5\not{s}}{2},\\ \hat{P}_3 &= \frac{-\not{p}+m}{2m}\frac{1-\gamma_5\not{s}}{2},\quad \hat{P}_4=\frac{-\not{p}+m}{2m}\frac{1+\gamma_5\not{s}}{2}.\end{aligned}$$

These projection operators are lorentz invariant, i.e. they are the same in every inertial frame! We are now viewed a special case, where $|\vec{p}|\ll m$ (ultra-relativistic scenario). We already know that in this case $\hat{\not{s}}\to\frac{\hat{\not{p}}}{m}$. In this case also the projection operators of the *helicity*, i.e.

$$\frac{1 \pm \gamma_5 \not{s}}{2},$$

and of the *chirality*, i.e.

$$\frac{1 \pm \gamma_5}{2},$$

are the same. In both cases the $-$ operator projects on left-handed particles, whereas the $+$ operator projects on right-handed particles.

1.3 Bilinear forms and $\hat{\mathcal{P}}$, $\hat{\mathcal{C}}$ and $\hat{\mathcal{T}}$

We have found, that $\overline{\psi}(x)\psi(x)$ is a Lorentz scalar, thus being invariant. Now we are interested in the classification in general:

- As we already know we get one scalar from

$$\overline{\psi}(x)\psi(x).$$

- Having an arbitrary γ matrix between gives us four vectors

$$\overline{\psi}(x)\gamma_\mu\psi(x).$$

- We have six choices for some matrix between our spinor,

$$\overline{\psi}(x)\sigma_{\mu\nu}\psi(x).$$

- By inserting a γ matrix multiplied with γ_5 we have another 4 possibilities (called axial vector or pseudo vector)

$$\overline{\psi}(x)\gamma_\mu\gamma_5\psi(x).$$

- And so we can also just insert a γ_5 matrix, which is one pseudo scalar, i.e.

$$\overline{\psi}(x)\gamma_5\psi(x).$$

We will now look at the different sub groups of lorentz transformations.

1.3.1 The charge-conjugation $\hat{\mathcal{C}}$

We define a transformation that

$$\left[i\gamma_\mu\partial^\mu - e\not{A} - m\right]\psi = 0$$

goes over into

$$\left[i\gamma_\mu\partial^\mu + e\not{A} - m\right]\psi_C = 0. \tag{1.5}$$

We already see that this has something to do with complex conjugation. By just performing a complex conjugation we get

$$\left[-i\gamma_\mu^*\partial^\mu - eA^\mu\gamma_\mu^* - m\right]\psi^* = 0.$$

To get a satisfying result we now put the ansatz,

$$\psi_C(x) = C\psi^*(x),$$

into equation (1.5) and do a complex conjugation to get rid of our $\psi^*(x)$. We obtain

$$\left[-i\gamma_\mu^*\partial^\mu + e\gamma_\mu^*A^\mu - m\right]C^*\psi(x) = 0.$$

Multiplying from left with $(C^*)^{-1}$ gives us

$$\left[-i(C^*)^{-1}\gamma_\mu^*C^*\partial^\mu + e\underbrace{(C^*)^{-1}\gamma_\mu^*C^*}_{\stackrel{!}{=}-\gamma_\mu}A^\mu - m\right]\psi(x) = 0.$$

By knowing that $\gamma_0^* = \gamma_0$ as γ_1 and γ_3 we can see that

$$C = \gamma_2\exp(i\varphi),$$

where φ is some arbitrary phase. So finally we have our operator,

$$\hat{\mathcal{C}} = \gamma_2\exp(i\varphi)\text{c.c.}, \tag{1.6}$$

where c.c. is the complex conjugation. By remembering that

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & x & 0 \\ 0 & x & 0 & 0 \\ x & 0 & 0 & 0 \end{pmatrix},$$

we see that this operator will exchange the upper and lower elements. Thus E and $\vec{p}$ will switch the sign. We see that

$$\psi_C = C\psi^* \propto \exp(+ipx).$$

Now the energy E_c is positiv. By saying that an antiparticle is a particle which has negative energy and runs back in time, it is just meant that

$$\exp(-iE_c t) = \exp(-i(-E_c)(-t)).$$

1.3.2 The parity $\hat{\mathcal{P}}$

We already know that $\vec{x} \to -\vec{x}$. We will perform the same ansatz as before,

$$\psi_p(x') = \psi_p(x^0, -\vec{x}) = \hat{\mathcal{P}}\psi(x).$$

Before solving the problem

$$\left[i\gamma^\mu \frac{\partial}{\partial x'^\mu} - e\gamma^\mu A_\mu(x') - m\right]\psi_p(x') = 0,$$

we have to remember something from classical electrodynamics.

Remark We already know that $\vec{E} \to -\vec{E}$ and $\vec{B} \to \vec{B}$ under parity. So our four vector potential A has to be $A^\mu \to (A^0, -\vec{A})$ in order to fulfill

$$-\frac{\partial}{\partial t}(-\vec{A}) - (-\nabla)A^0 = -\vec{E}, \qquad (-\nabla) \times (-\vec{A}) = \vec{B}.$$

With this knowledge we have

$$\left[i\gamma^0\frac{\partial}{\partial^0} - i\gamma^j\frac{\partial}{\partial^j} - e\gamma^0 A_0 + e\gamma^j A_j - m\right]\hat{\mathcal{P}}\psi(x) = 0.$$

In order to commutate with γ^0 and anticommutate with γ^j we can find γ_0. Totally we now know that

$$\hat{\mathcal{P}} = \gamma_0 \exp(i\chi). \tag{1.7}$$

1.3.3 The time-reversal $\hat{\mathcal{T}}$

In this case we first think of the Maxwell equations,

$$\begin{aligned} \nabla\vec{E} &= 4\pi\varrho \xrightarrow{\hat{\mathcal{T}}} 4\pi\varrho \quad \Rightarrow \quad \vec{E}\to\vec{E}, \\ \nabla\times\vec{B} &= 4\pi\vec{j} \xrightarrow{\hat{\mathcal{T}}} -4\pi\vec{j} \quad \Rightarrow \quad \vec{B}\to -\vec{B}. \end{aligned}$$

So we have found that $A'^j = -A^j$, which gives us

$$\left[i\gamma^\mu\frac{\partial}{\partial x'^\mu} - e\gamma^\mu A'^j_\mu(x') - m\right]\psi_T(x') = 0.$$

By following the formalism from the charge-conjugation we finally arrive at

$$\left[-i\gamma^0\frac{\partial}{\partial x^0} + i\gamma^j\frac{\partial}{\partial x^j} - e\gamma^0 A_0(x) + e\gamma^j A_j(x) - m\right]\psi_T(x') = 0,$$

which can be solved by using the ansatz

$$\psi_T(x') = T\psi^*(x),$$

as in the charge-conjugation case. So we multiply from left with $(T^*)^{-1}$ in order to obtain

$$\left[(T^*)^{-1}\gamma^0 T^*(i\frac{\partial}{\partial x^0} - eA_0(x)) - (T^*)^{-1}\gamma^{j*}T^*(i\frac{\partial}{\partial x^j} - eA_j(x)) - m\right]\psi(x) = 0.$$

Since T^* commutates with γ_0 and γ_2 and anticommutates with γ_1 and γ_3 we finally know the result, that

$$\hat{\mathcal{T}} = \gamma^1\gamma^3 \exp(i\xi)\text{c.c.}, \tag{1.8}$$

with some arbitrary phase ξ, which gives us

$$\psi_T(x') = \exp(i\xi)\gamma^1\gamma^3\psi^*(x).$$

1.4 The QED as gauge group

The whole QED can be derived from the demand, that physical objects are invariant under local phase transformations of the Dirac fields. That means

$$\psi(x) \longrightarrow \exp(iq\theta(x))\psi(x)$$

leaves all bilinear forms invariant. The lagrange density of the Dirac theory is

$$\mathcal{L}(x) = \overline{\psi}(x)\left[i\gamma^\mu\partial_\mu - m\right]\psi(x).$$

Therefore we have the equation of motion

$$\Rightarrow \frac{\delta\int d^4x'\mathcal{L}(x')}{\delta\overline{\psi}(x)} \Rightarrow \frac{\partial\mathcal{L}(x)}{\partial\overline{\psi}(x)} = \left[i\gamma^\mu\partial_\mu - m\right]\psi(x).$$

If we now insert the gauge parameter we see that

$$\mathcal{L} \to \overline{\psi}(x)\left[i\gamma^\mu\partial_\mu - q\gamma^\mu\left(\partial_\mu\theta(x)\right) - m\right]\psi(x)$$

is not invariant!

Thus we postulate a new field which scatters the additional term. Since $\partial_\mu\theta(x)$ is a lorentz vector and a field, we need a vector field $A_\mu(x)$. So we have

$$\mathcal{L}' = \overline{\psi}(x)\left[i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu(x) - m\right]\psi(x),$$

with the gauge transformation $A_\mu(x) \to A_\mu(x) - \partial_\mu\theta(x)$. We see that this is invariant.

Definition We now define

$$i\hat{D}_\mu \equiv i\hat{\partial}_\mu - qA_\mu(x)$$

as *gauge-invariant* derivative.

How can the dynamic of the new field look like? We search for an answer without making new assumptions. We already know, that

$$S = \int d^4x \mathcal{L}_{\mathrm{QED}}.$$

This is a number ($\hbar = c = 1$) where d^4x has dimension length^4 or energy^{-4}. Thus $\mathcal{L}_{\mathrm{QED}}$ must have dimension energy^4. Therefore $\mathcal{L}_{\mathrm{QED}}$ has an additional term which has 4 factors $\hat{D}_\mu$. Furthermore $\mathcal{L}_{\mathrm{QED}}$ is no operator, but $\hat{D}_\mu$ is a derivative operator. So we look at the commutator,

$$\begin{aligned}
\left[\hat{D}_\mu, \hat{D}_\nu\right] &= [\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] = \\
&= \partial_\mu\partial_\nu + iq(\partial_\mu A_\nu) + iqA_\nu\partial_\mu - q^2 A_\mu A_\nu + iqA_\mu\partial_\nu - \\
&- (\partial_\nu\partial_\mu + iq(\partial_\nu A_\mu) + iqA_\mu\partial_\nu - q^2 A_\nu A_\mu + iqA_\nu\partial_\mu) = \\
&= iqF_{\mu\nu},
\end{aligned}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and has dimension energy^2. So finally we have the additional term, which describes the dynamic of the $\vec{A}$ field,

$$\mathcal{L}_{\mathrm{QED}} = ... - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x),$$

where $\frac{1}{4}$ is an agreement which is based on the charge definition. This brings us to the lagrange density of the QED,

$$\boxed{\mathcal{L}_{\mathrm{QED}} = \sum_{\text{fermions } j} \overline{\psi}_j(x) \left(\hat{\not{p}} - eQ_j \not{A}(x) - m_j\right) \psi_j(x) - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x). \qquad (1.9)}$$

2 The Feynman propagator

First of all we have to remember what a Green's function is. Suppose we have an equation

$$\left(i\frac{\partial}{\partial t} - \hat{H}_0\right)\varphi(x) = j(x).$$

So we already know a formal solution in the form of

$$\begin{aligned}
\varphi(x) &= \int d^4x G(x,x')j(x'), \\
\delta^4(x-x') &= \left(i\frac{\partial}{\partial t} - \hat{H}_{0,x}\right)G(x,x').
\end{aligned}$$

By inserting this into the equation we see that this is a solution,

$$\left(i\frac{\partial}{\partial t} - \hat{H}_0\right) = \int d^4x' \left(i\frac{\partial}{\partial t} - \hat{H}_0\right) G(x,x')j(x') = \int d^4x'\delta^4(x-x')j(x') = j(x).$$

An application to this would be electrodynamics with Dirichlet boundary conditions,

$$\begin{aligned}
\nabla^2 G_D(\vec{x},\vec{x}') &= 4\pi\delta^3(\vec{x}-\vec{x}'), \\
\phi(x) &= \int d^3x' \varrho(\vec{x}')G_D(\vec{x},\vec{x}').
\end{aligned}$$

We now look at the Green's function of the Dirac theory. It is enough to determine the Green's function of the Klein-Gordon equation, therefore we have

$$\begin{aligned}
\left(\Box_x^2 + m^2\right)G_{KG}(x,x') &= -\delta^4(x-x'), \\
S_D(x,x') &= \left(i\partial_\mu\gamma^\mu + m\right)G_{KG}, \\
\Rightarrow \left(i\partial_\mu\gamma^\mu - m\right)S_D(x,x') &= \left(-\Box_x^2 - m^2\right)G_{KG}(x,x') = \delta^4(x-x').
\end{aligned}$$

Now we have to look at explicit presentation of the $\delta(x)$ and $\Theta(x)$ distributions:

$$\begin{aligned}\Theta(x) &= \lim_{\varepsilon\to 0}\frac{-1}{2\pi i}\int_{-\infty}^{\infty}dp\frac{\exp(-ipx)}{p+i\varepsilon} = \\ &= \begin{cases} x>0: & \lim_{\varepsilon\to 0}\frac{-1}{2\pi i}\int dp\frac{\exp(-ipx)}{p+i\varepsilon}=1 \\ x<0: & \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int dp\frac{\exp(-ipx)}{p+i\varepsilon}=0 \end{cases}.\end{aligned}$$

Definition of the δ distribution:

$$\delta(x) = \frac{d}{dx}\theta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dp\,\exp(-ipx).$$

The calculation of G_{KG} is quite easy in momentum space. Thus we obtain

$$G(x-x') = \int\frac{d^4p}{(2\pi)^4}\exp(-ip(x-x'))G_{KG}(p).$$

This is a consequence of translation invariance, because we have no external fields, so

$$G_{KG}(x,x') = G_{KG}(x-x').$$

We see directly that

$$\int\frac{d^4p}{(2\pi)^4}(\hat{p}^2-m^2)\exp(-ipx)G_{KG}(p) = \int\frac{d^4p}{(2\pi)^4}\exp(-ipx) = \delta^4(x).$$

The problem is now that

$$(p^2-m^2)G_{KG}(p) = 1$$

has no well defined reverse transformation,

$$\int\frac{d^4p}{(2\pi)^4}\frac{\exp(-ip(x-x'))}{p^2-m^2}.$$

The reason for this is that we have to integrate over the poles $p^2=m^2$. Therefore we require a rule how to pass the poles correctly. This is equivalent to the rule how we can move the poles by an ε. We have four possible choices:

1. Both poles above the real axis.

2. The first one $(-\sqrt{\vec{p}^2+m^2})$ below, the second one above.
3. The first one above and the second one $(\sqrt{\vec{p}^2+m^2})$ below.
4. Both poles below the real axis.

With two infinitesimal values $\varepsilon, \eta > 0$ we obtain

$$\begin{aligned}
(1) \quad &\Rightarrow \quad \frac{1}{p^2 - m^2 - i\varepsilon \mathrm{sgn}(\mathrm{Re}(p^0))} \\
&\Rightarrow \quad p^0 = \pm\sqrt{\vec{p}^2+m^2} + i\eta, \qquad (p^0)^2 = \vec{p}^2 + m^2 + \mathrm{sgn}(p^0) i\varepsilon, \\
(2) \quad &\Rightarrow \quad \frac{1}{p^2 - m^2 - i\varepsilon} \\
&\Rightarrow \quad p^0 = \pm\sqrt{\vec{p}^2+m^2} \pm i\eta, \qquad (p^0)^2 = \vec{p}^2 + m^2 +) i\varepsilon, \\
(3) \quad &\Rightarrow \quad \frac{1}{p^2 - m^2 + i\varepsilon} \\
&\Rightarrow \quad p^0 = \pm\sqrt{\vec{p}^2+m^2} \pm i\eta, \qquad (p^0)^2 = \vec{p}^2 + m^2 - i\varepsilon, \\
(4) \quad &\Rightarrow \quad \frac{1}{p^2 - m^2 + i\varepsilon \mathrm{sgn}(\mathrm{Re}(p^0))} \\
&\Rightarrow \quad p^0 = \pm\sqrt{\vec{p}^2+m^2} - i\eta, \qquad (p^0)^2 = \vec{p}^2 + m^2 - \mathrm{sgn}(p^0) i\varepsilon.
\end{aligned}$$

We demand that for $\mathrm{Re}(p^0) < 0$ $G(x-x')$ must be zero for $t > t'$. We also demand that for $\mathrm{Re}(p^0 >)0$ $G(x-x') = 0$ for $t < t'$. With $t > t'$ we now look at

$$\begin{aligned}
G(x-x') &= \int_{-\infty}^{\infty} dp^0 \int \frac{d^3p}{(2\pi)^4} \exp(-ip^0(t-t') + i\vec{p}(\vec{x}-\vec{x}')) G(p^0, \vec{p}) = \\
&= \int \frac{d^3p}{(2\pi)^4} \int dp^0 \ldots.
\end{aligned}$$

We will integrate by running over the real axis and closing the half circle below the real axis. By doing the same for $t < t'$ with closing the half circle above the real axis, we see that the only solution for this can be the 3rd way (3). Therefore we found the Feynman propagator $G_F(p)$ in momentum space,

$$G_F(p) = \frac{1}{p^2 - m^2 + i\varepsilon}. \tag{2.1}$$

After some calculation we also find the Feynman propagator $G_F(x)$ in real space,

$$G_F(x) = -\frac{1}{4\pi}\delta(x^2) + \frac{m}{8\pi\sqrt{x^2}}\Theta(x^2)\left[J_1(m\sqrt{x^2}) - iN_1(m\sqrt{x^2})\right] - \qquad (2.2)$$
$$- \frac{im}{4\pi\sqrt{-x^2}}\Theta(-x^2)K_1(m\sqrt{-x^2}).$$

In this equation J_ν are Bessel functions of the first kind, N_ν are Bessel functions of the second kind (also called Neumann functions) and K_ν are modified Bessel functions. This result is physically totally understandable, when we think of how

$$|\vec{x}|^2 - (x^0)^2 = 0$$

defines the light cone. We directly see that $\delta(x^2)$ defines the boundaries of the light cone and that J_1 and N_1 are just oscillating functions. In $K_1 \propto \exp(-m\sqrt{-x^2})$ we have to decay which is described by the Compton wave-length like in quantum mechanics.

We will now look at a different form for the propagator, based on our projection operators. We know that (in the rest frame)

$$u(\vec{p},+)\overline{u}(\vec{p},+) + u(\vec{p},-)\overline{u}(\vec{p},-) = \Lambda_+ = \frac{\not{p}+m}{2m} \Rightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Due to the boost invariance of $\int d^3xE$ we have to define other operators in moving systems. We get

$$\begin{aligned} \Lambda_1 &= 2E\psi_1(x)\overline{\psi}_1(x), \\ \Lambda_2 &= 2E\psi_2(x)\overline{\psi}_2(x), \\ \text{e.g. } \chi(x) &= \int d^3x 2E\psi(x')\overline{\psi}(x)\chi(x). \end{aligned}$$

We will now calculate

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\exp(-ip(x-y))\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\left[\Theta(x^0-y^0)+\Theta(y^0-x^0)\right].$$

By using the residue theorem and $p^0 = \pm E = \pm\sqrt{\vec{p}^2+m^2}$ we obtain

$$
\begin{aligned}
S_F(x-y) &= -i\int \frac{d^3p}{(2\pi)^3}\exp(-iE(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y}))\frac{\overbrace{E\gamma^0-\vec{p}\vec{\gamma}}^{\not p}+m}{2E}\Theta(x^0-y^0)+ \\
&+ i\int \frac{d^3p}{(2\pi)^3}\exp(iE(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y}))\frac{-E\gamma^0-\vec{p}\vec{\gamma}+m}{-2E}\Theta(y^0-x^0)= \\
&= -i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E}\exp(-iEx^0+i\vec{p}\vec{x})(\not p+m)\exp(iEy^0+i\vec{p}\vec{y})\Theta(x^0-y^0)+ \\
&+ i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E}\underbrace{\exp(iEx^0-i\vec{p}\vec{x})(\not p-m)\exp(-iEy^0+i\vec{p}\vec{y})}_{2E\Lambda_\pm(x,y)}\Theta(y^0-x^0)= \\
&= -i\int \frac{d^3p}{(2\pi)^3}\left(\sum_{\nu=1,2}\psi_\nu(p,x)\overline{\psi}_\nu(p,y)\Theta(x^0-y^0)-\sum_{\nu=3,4}\psi_\nu(p,x)\overline{\psi}_\nu(p,y)\Theta(y^0-x^0)\right).
\end{aligned}
$$

Remark Normalization of the plain waves,

$$\psi = Nu(\vec{p},t)\exp(-ipx).$$

We calculate

$$
\begin{aligned}
I &= \int d^3x\overline{\psi}(\vec{p},\vec{y},t)\psi(\vec{p},\vec{x},t) = |N|^2\int d^3x\exp(i\vec{p}(\vec{x}-\vec{y})-iE(t-t))\underbrace{\overline{u})(\vec{p},+)u(\vec{p},+)}_{=1} = \\
&= |N|^2(2\pi)^3\delta^3(p).
\end{aligned}
$$

Now we can calculate the limit $y\to x$. Since we always have to deal with integrals $\int d^3p$, e.g. proof of a particle in a collision experiment, where we have to integrate over the $(\Delta p)^3$ of the detector, we have always calculations like

$$\int\frac{d^3p}{(2\pi)^3}\delta^3(\vec{p})X = Y,$$

where X is not lorentz-invariant in general, but Y is always lorentz-invariant. Thus our task is to add something so that the left side is also lorentz-invariant in general. We see that

$$\int\frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)2E\frac{1}{2E}...$$

with $p^0 = \pm\sqrt{\vec{p}^2 + m^2} = \pm E$ is lorentz-invariant. Thus we replace

$$\begin{aligned}\tilde{u}(p,+) &= \frac{\sqrt{2m}}{\sqrt{2E}}u(p,+), \\ \tilde{\psi} &= \sqrt{\frac{2m}{2E}}\psi.\end{aligned}$$

So now we see that $\tilde{\overline{u}}\tilde{u} = \frac{2m}{2E}$. This gives us

$$\sum_s \tilde{u}(p,s)\overline{\tilde{u}}(p,s) = \frac{2m}{2E}\frac{\not{p}+m}{2m} = \frac{\not{p}+m}{2E},$$

i.e. the projection operator on positive energy is

$$\tilde{\Lambda}_+ = \not{p} + m = 2E\sum_s \tilde{u}(p,s)\overline{\tilde{u}}(p,s).$$

- Other normalization factors are like $\frac{1}{\sqrt{2E}}u(p,+)$. The advantage of our normalization factor compared to factors of this type is that

$$\lim_{m\to 0}\frac{\not{p}+m}{2E}$$

  exists.

- Another possible choice would be $\sqrt{\frac{m}{EV}}u(p,+)$ with the quantization volume $V = L^3$, so that

$$(2\pi)^3\delta^3(\vec{p}) = \int_{-L/2}^{L/2} d^3x \exp(i\vec{p}\vec{x}) = V, \qquad p_i = \frac{2\pi}{L}n, n \in \mathbb{Z}.$$

  We will use that factor for the solution of a certain problem later on.

3 The relativistic hydrogen atom

First of all we have to see that in the relativistic case $\vec{l}$ is no conserved quanitity!

- Non-relativistic Schrödinger equation

$$\hat{H}_S\psi = E\psi, \qquad \hat{H}_S = \frac{\hat{\vec{p}}^2}{2m} + eV(r).$$

  We can use $V(\vec{r}) \equiv V(r)$, because of the Coulomb-Potential (central force). We already know that

$$\vec{l} = \hat{\vec{r}} \times \hat{\vec{p}}, \qquad \frac{d\vec{l}}{dt} = \underbrace{\frac{\partial \vec{l}}{\partial t}}_{=0} + [\hat{\vec{l}}, \hat{H}_S].$$

  We analyse the commutator and see that

  - For the first term we have

$$[\hat{\vec{r}} \times \hat{\vec{p}}, \hat{\vec{p}}^2]_k \propto -2\varepsilon_{klm}\hat{p}^j \underbrace{[\hat{p}^j, \hat{r}^l]}_{-i\delta_{jl}} \hat{p}^m \propto \varepsilon_{klm}\hat{p}^l\hat{p}^m = 0.$$

  - For the second term we calculate

$$[\hat{\vec{r}} \times \hat{\vec{p}}, V(r)] \propto \vec{r} \times [\hat{\vec{p}}V] \propto \vec{r} \times \vec{r} = 0.$$

  Summary: In the non-relativistic scenario is $\vec{l}$ a conserved quantity.

- In the relativistic Dirac equation $(\not{p} - e\not{A} - m)\psi = 0$ we have

$$\hat{p}^0\gamma_0 = \hat{H}_D\gamma_0 \qquad \hat{H}_D = (\hat{\vec{p}}\vec{\gamma} + m)\gamma_0 + eV.$$

  So in this scenario we calculate

$$[\vec{r}\times\hat{\vec{p}},\hat{\vec{p}}\vec{\gamma}\gamma_0]_k=\varepsilon_{klm}\hat{p}^m[\vec{r},\hat{p}^n]\gamma^n\gamma_0=i\varepsilon_{klm}\hat{p}^m\gamma^l\gamma_0\neq 0!$$

The spin is not conserved as well. By using

$$\vec{\Sigma}\equiv\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix},$$

we can calculate

$$\begin{aligned}
[\vec{\Sigma},\hat{H}_D] &= \hat{p}^m[\vec{\Sigma},\underbrace{\gamma^m\gamma_0}_{=-\begin{pmatrix}0 & \sigma^m\\ \sigma^m & 0\end{pmatrix}}]=\\
&= -\hat{p}^m\left[\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix},\begin{pmatrix}0 & \sigma^m\\ \sigma^m & 0\end{pmatrix}\right]=\\
&= -\hat{p}^m\begin{pmatrix}0 & \sigma^k\sigma^m-\sigma^m\sigma^k\\ \sigma^k\sigma^m-\sigma^m\sigma^k & 0\end{pmatrix}=\\
&= -\hat{p}^m 2i\varepsilon_{kml}\begin{pmatrix}0 & \sigma^l\\ \sigma^l & 0\end{pmatrix}=-2i\hat{p}^m\varepsilon_{klm}\gamma^l\gamma_0.
\end{aligned}$$

Therefore we found another conserved quantity by setting

$$\vec{J}=\vec{l}+\frac{1}{2}\vec{\Sigma}. \tag{3.1}$$

We call $\vec{J}$ the total angular momentum.

In the relativistic scenario we can also find another conserved quantity,

$$\hat{\kappa}\equiv\gamma^0(\vec{\Sigma}\vec{l}+1).$$

Proof We will calculate the commutator in order to see that κ is a conserved quantity.

$$\begin{aligned}
[\hat{\kappa},\hat{H}_D] &= \gamma^0\Sigma^k[l^k,\hat{H}_D]+\gamma^0[\Sigma^k,\hat{H}_D]l^k+[\gamma^0,\hat{H}_D]\vec{\Sigma}\vec{l}+[\gamma^0,\hat{H}_D]=\\
&= \gamma^0\Sigma^k i\varepsilon_{klm}\hat{p}^m\gamma^l\gamma_0-2i\gamma^0\varepsilon_{klm}\hat{p}^m\gamma^l\gamma_0 l^k-2\hat{\vec{p}}\vec{\gamma}\vec{\Sigma}\vec{l}-2\hat{\vec{p}}\vec{\gamma}.
\end{aligned}$$

With

$$\varepsilon_{klm}\Sigma^k\gamma^l\gamma_0 = 2i\gamma^m\gamma_0$$

we see that the first term will transform to

$$i\gamma^0 2i\gamma^m\gamma_0 p^m = -2i\gamma^m p^m i = 2p^m\gamma^m$$

and vanishes the 4th term. The third term becomes

$$-2\begin{pmatrix} 0 & \vec{p}\vec{\sigma} \\ -\vec{p}\vec{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \vec{\sigma}\vec{l} & 0 \\ 0 & \vec{\sigma}\vec{l} \end{pmatrix} = ... = -2i\varepsilon_{mkl}\hat{p}^m\hat{l}^k\gamma^l$$

and vanishes the 2nd term. Thus we have shown that

$$[\hat{\kappa}, \hat{H}_D] = 0,$$

which proofes that κ is indeed a conserved quantity.

We will show now that $\hat{\kappa}$ and $\hat{j}^i$ commutate. We already know that

$$\hat{j}^i = \hat{l}^i + \frac{1}{2}\hat{\Sigma}^i, \qquad \hat{\Sigma}^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$$

Therefore we calculate

$$\begin{aligned} [\gamma^0\hat{\vec{\Sigma}}\vec{l}, \vec{l}^i] + [\gamma^0\hat{\vec{\Sigma}}\vec{l}, \frac{1}{2}\hat{\Sigma}^i] &= \gamma^0\hat{\Sigma}^k[\hat{l}^k, \hat{l}^i] + \frac{1}{2}\gamma^0[\hat{\Sigma}^k, \hat{\Sigma}^i]\hat{l}^k = \\ &= \gamma^0\hat{\Sigma}^k i\varepsilon_{kil}\hat{l}^l + \frac{1}{2}\gamma^0 2i\varepsilon_{kil}\hat{\Sigma}^l\hat{l}^k = 0. \end{aligned}$$

Thus $\hat{\kappa}$ commutates with $\hat{\vec{j}}$ (and $\hat{\vec{j}}^2$ as well). Finally with the general relation

$$\vec{\sigma}\hat{\vec{A}}\vec{\sigma}\hat{\vec{B}} = \sigma^i\sigma^j\hat{A}^i\hat{B}^j = \hat{\vec{A}}\hat{\vec{B}} + i\varepsilon_{klm}\sigma^k\hat{A}^l\hat{B}^m,$$

we see that

$$
\begin{aligned}
\hat{\kappa}^2 &= \left(\hat{\vec{\Sigma}}\hat{\vec{l}}+1\right)\left(\hat{\vec{\Sigma}}\hat{\vec{l}}+1\right) = \begin{pmatrix} \vec{\sigma}\hat{\vec{l}}\vec{\sigma}\hat{\vec{l}} & 0 \\ 0 & \vec{\sigma}\hat{\vec{l}}\vec{\sigma}\hat{\vec{l}} \end{pmatrix} + 2\hat{\vec{\Sigma}}\hat{\vec{l}} + 1, \\
&= \begin{pmatrix} \hat{\vec{l}}^2 + i\varepsilon_{klm}\sigma^k\hat{l}^l\hat{l}^m & 0 \\ 0 & \hat{\vec{l}}^2 + \frac{1}{2}i\underbrace{\varepsilon_{klm}i\varepsilon_{\varrho lm}}_{=2i\delta_{k\varrho}}\hat{l}^\varrho\sigma^k \end{pmatrix} + 2\hat{\vec{\Sigma}}\hat{\vec{l}} + 1 = \\
&= \underbrace{\begin{pmatrix} \hat{\vec{l}}^2 - \vec{\sigma}\hat{\vec{l}} & 0 \\ 0 & \hat{\vec{l}}^2 - \vec{\sigma}\hat{\vec{l}} \end{pmatrix}}_{=\hat{\vec{l}}^2-\hat{\vec{\Sigma}}\hat{\vec{l}}} + 2\hat{\vec{\Sigma}}\hat{\vec{l}} + 1 = \\
&= \left(\hat{\vec{l}} + \frac{1}{2}\hat{\vec{\Sigma}}\right)^2 - \frac{1}{4}\hat{\vec{\Sigma}}^2 + 1.
\end{aligned}
$$

With $\vec{\Sigma}^2 = 3$ follows directly that

$$
\hat{\kappa}^2 = \hat{\vec{j}}^2 + \frac{1}{4}.
$$

We define the eigenvalues of $\hat{\kappa}\psi$ with $-\kappa\psi$. Therefore we get for the eigenvalues

$$
\kappa^2 = j(j+1) + \frac{1}{4}. \tag{3.2}
$$

If κ is an eigenvalue, then $-\kappa$ is an eigenvalue as well. We now look at the eigenvalues for $j = n/2$. We see that

$$
\kappa^2 = \frac{n(n+2)}{4} + \frac{1}{4} = \frac{n^2}{4} + \frac{n}{2} + \frac{1}{4} = \left(\frac{n}{2} + \frac{1}{2}\right)^2.
$$

That means that $|\kappa| = j + \frac{1}{2}$ gives us j. We will now investigate the sign of κ. This follows from

$$
\hat{\kappa} = \begin{pmatrix} \vec{\sigma}\hat{\vec{l}} + 1_{2\times 2} & 0 \\ 0 & -(\vec{\sigma}\hat{\vec{l}} + 1_{2\times 2}) \end{pmatrix}.
$$

We have

$$
\vec{\sigma}\hat{\vec{l}} = \left(\frac{1}{2}\vec{\sigma} + \hat{\vec{l}}\right)^2 - \left(\frac{\vec{\sigma}}{2}\right)^2 - \left(\hat{\vec{l}}\right)^2 = \hat{\vec{j}}^2 - \left(\frac{\vec{\sigma}}{2}\right)^2 - \left(\hat{\vec{l}}\right)^2.
$$

So we get for the eigenvalues

$$j(j+1) - \frac{1}{2}\frac{3}{2} - l(l+1).$$

Thus we have to possibilities,

1. $j = l + \frac{1}{2}$. So the eigenvalue is

$$\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right) - \frac{3}{4} - l(l+1) = l \geq 0.$$

   Since the eigenvalue is $-\kappa - 1$ we found $\kappa = -l - 1$.

2. $j = l - \frac{1}{2}$. So the eigenvalue is

$$\left(l+\frac{1}{2}\right)\left(l-\frac{1}{2}\right) - \frac{3}{4} - l(l+1) = -l - 1 < 0.$$

   Since the eigenvalue is $-\kappa - 1$ we found $\kappa = l$.

We conclude that if κ is positiv we have $l = \kappa$ and $j = \kappa - \frac{1}{2}$, while if κ is negativ we have $l = -\kappa - 1$ and $j = -\kappa - \frac{1}{2}$.

κ	0	-1	1	-2	2	...
j	-	1/2	1/2	3/2	3/2	...
l	-	0	1	1	2	...
		$s_{1/2}$	$p_{1/2}$	$p_{3/2}$	$d_{3/2}$	...

General we can say that $l = j + \text{sign}(\kappa) \cdot \frac{1}{2}$. The eigenvalues and eigenstates (e.g.) of κ are

$$\begin{pmatrix} -\kappa 1_{2\times 2} & 0 \\ 0 & \kappa 1_{2\times 2} \end{pmatrix}, \qquad \begin{pmatrix} \psi_{2s_{1/2}} \\ \psi_{2p_{1/2}} \end{pmatrix}.$$

The upper and lower double spinors are equivalent to the solutions of the Pauli equation. The eigenstates are equivalent to the basis of the Pauli equation,

$$|jls\mu\rangle = \sum_{m,s_z} (lmss_z|j\mu) Y_{lm}(\vartheta,\varphi)\chi_{s_z}, \qquad \left\{ \begin{array}{ccc} \chi_{1/2} & = & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \chi_{-1/2} & = & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array} \right. .$$

For the Clebsch-Gordon coefficants we calculate

$$\left(l\left(\mu+\frac{1}{2}\right)\frac{1}{2}\left(\mp\frac{1}{2}\right)|j\mu\right),$$

and find that:

	$j=l+\frac{1}{2}$	$j=l-\frac{1}{2}$
$l=\mu+\frac{1}{2},\ s_z=-\frac{1}{2}$	$\sqrt{\frac{l-\mu+\frac{1}{2}}{2l+1}}$	$\sqrt{\frac{k+\mu+\frac{1}{2}}{2l+1}}$
$l=\mu-\frac{1}{2},\ s_z=\frac{1}{2}$	$\sqrt{\frac{l+\mu+\frac{1}{2}}{2l+1}}$	$-\sqrt{\frac{k-\mu+\frac{1}{2}}{2l+1}}$

Now we define the basis states of the Pauli equation for $l=j\mp\frac{1}{2}$,

$$\varphi^{\pm}_{j\mu}(x)=\pm\left(\begin{array}{c}\pm\sqrt{\frac{l\pm\mu+\frac{1}{2}}{2l+1}}Y_{l(\mu-\frac{1}{2})}(\vartheta,\varphi)\\ \sqrt{\frac{l\mp\mu+\frac{1}{2}}{2l+1}}Y_{l(\mu+\frac{1}{2})}(\vartheta,\varphi)\end{array}\right).$$

We are now making an ansatz in the form of

$$\psi_{j\mu}(x)=\exp(-iEx_0)\left(\begin{array}{c}(F_1(r)+iG_1(r))\varphi^{+}_{j\mu}+(F_2(r)+iG_2(r))\varphi^{-}_{j\mu}\\ (F_3(r)+iG_3(r))\varphi^{+}_{j\mu}+(F_4(r)+iG_4(r))\varphi^{-}_{j\mu}\end{array}\right).$$

<u>If</u> we can choose eigenfunctions of $\hat{\mathcal{P}}$ and $\hat{\mathcal{T}}$, we obtain the general ansatz

$$\psi_{j\mu}(x)=\exp(-iE_x0)\left(\begin{array}{c}ig(r)\chi^{\mu}_{\kappa}(\vartheta,\varphi)\\ f(r)\chi^{\mu}_{-\kappa}(\vartheta,\varphi)\end{array}\right),$$

with the '*spherical spinors*',

$$\chi^{\mu}_{\kappa}\equiv\begin{cases}\varphi^{+}_{j\mu}, & \text{if } \kappa<0,\\ -\varphi^{-}_{j\mu}, & \text{if } \kappa>0.\end{cases}$$

We will now try to work with the relation

$$\frac{\vec{r}}{r}\vec{\sigma}\chi^{\mu}_{\kappa}=\vec{e}_r\vec{\sigma}\chi^{\mu}_{\kappa}=-\chi^{\mu}_{-\kappa}(\vartheta,\varphi).$$

In order to do this we first have to modify the Dirac equation to

$$\hat{p}^0\psi=\left(\hat{\vec{p}}\vec{\gamma}\gamma_0+m\gamma_0+eV(r)\right)\psi=\hat{H}_D\psi.$$

Now we are going to search solutions $\hat{H}_D\psi = E\psi$. Our task is the derivation of the radial equation for $f(r)$, $g(r)$. The difficult term is

$$\hat{\vec{p}}\vec{\gamma}\gamma_0 = \hat{\vec{p}}^j \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix} = \begin{pmatrix} 0 & -\hat{\vec{p}}\vec{\sigma} \\ -\hat{\vec{p}}\vec{\sigma} & 0 \end{pmatrix}.$$

We will now look at

$$\vec{p}\vec{\sigma} = -i\vec{\sigma}\nabla = -i\vec{\sigma}\left[\vec{e}_r(\vec{e}_r\nabla) - \vec{e}_r \times (\vec{e}_r \times \nabla)\right],$$

because of the Grassmann identity, $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\vec{C}) - \vec{C}(\vec{A}\vec{B})$. Thus we obtain

$$-i\vec{\sigma}\nabla = -i\vec{\sigma}\vec{e}_r\frac{\partial}{\partial r} + i\vec{\sigma}\left(\vec{e}_r \times \frac{\vec{r} \times \nabla}{r}\right).$$

With $\hat{\vec{l}} = -i\hat{\vec{r}} \times \nabla$ we get

$$-i\vec{\sigma}\nabla = -i\sigma\vec{e}_r\frac{\partial}{\partial r} - \frac{\vec{\sigma}}{r}(\vec{e}_r \times \hat{\vec{l}}).$$

Now we use another useful relation,

$$\vec{\sigma}\vec{A}\vec{\sigma}\vec{A} = \vec{A}\vec{B} + i\vec{\sigma}(\vec{A} \times \vec{B}) \quad \Rightarrow \quad \vec{\sigma}\vec{r}\vec{\sigma}\hat{\vec{l}} = \underbrace{\vec{r}\hat{\vec{l}}}_{=0} + i\vec{\sigma}(\vec{r} \times \hat{\vec{l}}).$$

Therefore we have

$$\frac{\vec{\sigma}}{r}(\vec{e}_r \times \hat{\vec{l}}) = -\frac{i}{r^2}\vec{\sigma}\vec{r}\vec{\sigma}\hat{\vec{l}}.$$

With our definition of $\kappa_{4\times 4} \equiv \hat{\kappa}$ and $\vec{\Sigma}$ we obtain

$$\vec{\Sigma}\hat{\vec{l}} = \gamma^0\kappa_{4\times 4} - 1_{4\times 4}.$$

Thus we found

$$\begin{aligned} -\frac{i}{r}\gamma_5 \begin{pmatrix} \vec{\sigma}\vec{e}_r & 0 \\ 0 & \vec{\sigma}\vec{e}_r \end{pmatrix} (\gamma_0\kappa_{4\times 4} - 1_{4\times 4}) &= -i \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \frac{1}{r} \begin{pmatrix} i\vec{\sigma}(\vec{e}_r \times \vec{l}) & 0 \\ 0 & i\vec{\sigma}(\vec{e}_r \times \vec{l}) \end{pmatrix} = \\ &= \frac{1}{r} \begin{pmatrix} 0 & \vec{\sigma}(\vec{e}_r \times \vec{l}) \\ \vec{\sigma}(\vec{e}_r \times \vec{l}) & 0 \end{pmatrix}. \end{aligned}$$

Now we put everything together and get

$$\hat{\vec{p}}\vec{\gamma}\gamma_0 = \begin{pmatrix} 0 & i\vec{\sigma}\vec{e}_r \\ i\vec{\sigma}\vec{e}_r & 0 \end{pmatrix} \frac{\partial}{\partial r} - \frac{i}{r}\gamma_5 \begin{pmatrix} \vec{\sigma}\vec{e}_r & 0 \\ 0 & \vec{\sigma}\vec{e}_r \end{pmatrix} (\gamma_0 \kappa_{4\times 4} - 1_{4\times 4}).$$

We now define

$$\kappa_{4\times 4} = \begin{pmatrix} -\kappa_{2\times 2} & 0 \\ 0 & \kappa_{2\times 2} \end{pmatrix} \qquad \text{with } \kappa_{2\times 2}\chi^\mu_\kappa = \kappa\chi^\mu_\kappa.$$

Now we have to think back to

$$\psi(x) = \exp(-iEx^0) \begin{pmatrix} ig(r)\chi^\mu_\kappa \\ f(r)\chi^\mu_\kappa \end{pmatrix},$$

and write $\hat{H}_D$ as block matrix,

$$\hat{H}_D = \begin{pmatrix} eV + m & i\vec{\sigma}\vec{e}_r \left(\frac{\partial}{\partial r} + \frac{\kappa_{2\times 2} + 1_{2\times 2}}{r}\right) \\ i\vec{\sigma}\vec{e}_r \left(\frac{\partial}{\partial r} + \frac{\kappa_{2\times 2} - 1_{2\times 2}}{r}\right) & eV - m \end{pmatrix}.$$

We are now inserting this in the equation $\hat{H}_D\psi - E\psi = 0$ and get two equations. First we calculate

$$\begin{aligned} (eV + m)ig(r)\chi^\mu_\kappa + i\vec{\sigma}\vec{e}_r \left(\frac{\partial f}{\partial r}(r) + \frac{-\kappa + 1}{r} f(r)\right) \chi^\mu_{-\kappa} - ig(r)E\chi^\mu_\kappa &= 0, \\ (eV + m)ig(r)\chi^\mu_\kappa - i \left(\frac{\partial f}{\partial r}(r) + \frac{-\kappa + 1}{r} f(r)\right) \chi^\mu_\kappa - ig(r)E\chi^\mu_\kappa &= 0. \end{aligned}$$

Therefore we get

$$\frac{\partial}{\partial r} f(r) = \frac{\kappa - 1}{r} f(r) + (eV + m - E)g(r). \tag{3.3}$$

By doing the same for the lower component we archieve

$$\frac{\partial}{\partial r} g(r) = -\frac{\kappa + 1}{r} g(r) - (eV - m - E)f(r). \tag{3.4}$$

These two equations are the radial equations for the relativistic hydrogen atom. The solution is obtained by doing the same as with the non-relativistic hydrogen atom:

- Take a look at the asymptotic analysis of the inner bounds.
- Take a look at the asymptotic analysis of the outer bounds.
- Make a power series ansatz.
- Set a terminating condition.

We will now just look at the asymptotic analysis of the inner bounds. The problem is that $eV = -Z\alpha/r$. Therefore we get

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\kappa - 1}{r} f - \frac{Z\alpha}{r} g,\\ \frac{\partial g}{\partial r} &= \frac{-\kappa - 1}{r} g + \frac{Z\alpha}{r} f.\end{aligned}$$

Thus it seems reasonable to make the ansatz

$$g(r) = g_0 r^\gamma, \qquad f(r) = f_0 r^\gamma.$$

By doing this we obtain

$$\begin{aligned}\gamma f_0 &= (\kappa - 1) f_0 - Z\alpha g_0,\\ \gamma g_0 &= -(\kappa + 1) g_0 + Z\alpha f_0.\end{aligned}$$

By modifying this result into

$$\begin{aligned}(\gamma - \kappa + 1) f_0 &= -Z\alpha g_0,\\ (\gamma + \kappa + 1) g_0 &= Z\alpha f_0,\end{aligned}$$

we can multiply the two equations. We get

$$\left[(\gamma + 1)^2 - \kappa^2\right] f_0 g_0 = -(Z\alpha)^2 g_0 f_0, \qquad \gamma = \sqrt{\kappa^2 - (Z\alpha)^2} - 1.$$

We see that this is not defined for the $1s_{1/2}$ state ($\kappa^2 = 1$) with $Z\alpha > 1$. Is there an explanation why not everything is real?

Answer: Yes, there is an explanation. The Dirac equation does know about the possibility to make e^+e^--pair production. The best overview is given in the dirac sea picture. There we see that for $Z\alpha > 1$ the binding energy is greater than $2m_e c^2$. Thus we have pair production. The electron will be bound, the positron is flying away.

4 Canonical quantization

We say that a quantization is a method, which gives us the correct Green's function. We want to obtain

$$\begin{aligned}
(iS_F(x-y))_{jl} &= \ldots = \langle 0|T\left\{\hat{\psi}_j(x)\hat{\overline{\psi}}_l(y)\right\}|0\rangle, \\
iD_F(x-y) &= \ldots = \langle|T\left\{\hat{\Phi}(x)\hat{\Phi}^*(y)\right\}|0\rangle.
\end{aligned}$$

The function $T\{...\}$ gives us time-ordered products.

Definition The time ordered product for fermions is

$$T\left\{\hat{\psi}_j(x)\hat{\overline{\psi}}_l(y)\right\} \equiv \Theta(x^0-y^0)\hat{\psi}_j(x)\hat{\overline{\psi}}_l(y) - \Theta(y^0-x^0)\hat{\overline{\psi}}_l(y)\hat{\psi}_j(x).$$

For bosons we have

$$T\left\{\hat{\Phi}(x)\hat{\Phi}^*(y)\right\} \equiv \Theta(x^0-y^0)\hat{\Phi}(x)\hat{\Phi}^*(y) - \Theta(y^0-x^0)\hat{\Phi}^*(y)\hat{\Phi}(x).$$

The contained objects are defined as

$$\begin{aligned}
\hat{\psi}(x) &= \sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left[\hat{b}(\vec{p},s)u(\vec{p},s)\exp(-ipx) + \hat{d}^\dagger(\vec{p},s)v(\vec{p},s)\exp(ipx)\right], \\
\hat{\psi}^\dagger(x) &= \sum_{s=\pm}\int\frac{d^3p}{(2\pi)^3 2E_p}\left[\hat{b}^\dagger(\vec{p},s)\overline{u}(\vec{p},s)\gamma_0\exp(ipx) + \hat{d}(\vec{p},s)\overline{v}(\vec{p},s)\gamma_0\exp(-ipx)\right], \\
E_p &= \sqrt{\vec{p}^2+m^2}.
\end{aligned}$$

For the unknown objects we specify

$$
\begin{aligned}
1 &= \langle 0|0\rangle, \\
0 &= \hat{b}|0\rangle = \hat{d}|0\rangle, \quad \text{annihilation operator} \\
0 &= \langle 0|\hat{b}^\dagger = \langle 0|\hat{d}^\dagger,, \quad \text{creation operator} \\
\{\hat{b}(\vec{p},s), \hat{b}^\dagger(\vec{p}',s')\} &= \delta_{ss'}\delta^3(\vec{p}-\vec{p}')(2\pi)^3 2p^0, \\
\{\hat{d}(\vec{p},s), \hat{d}^\dagger(\vec{p}',s')\} &= \delta_{ss'}\delta^3(\vec{p}-\vec{p}')(2\pi)^3 2p^0.
\end{aligned}
$$

Proof We will now proof that this definition fits with our demands. So we calculate

$$
\begin{aligned}
& \langle 0|T\left\{\hat{\psi}_j(x)\hat{\overline{\psi}}_l(y)\right\}|0\rangle = \\
= \;& \Theta(x^0-y^0)\sum_{s,s'}\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p}\int\frac{d^3\vec{p}'}{(2\pi)^3 2E_{p'}}\langle 0|\hat{b}(\vec{p},s)\hat{b}^\dagger(\vec{p}',s')|0\rangle e^{-ipx+ip'y}u_j(\vec{p},s)\overline{u}_l(\vec{p}',s') - \\
- \;& \Theta(y^0-x^0)\sum_{s,s'}\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p}\int\frac{d^3\vec{p}'}{(2\pi)^3 2E_{p'}}\langle 0|\hat{d}(\vec{p},s)\hat{d}^\dagger(\vec{p}',s')|0\rangle e^{ipx-ip'y}v_j(\vec{p},s)\overline{v}_l(\vec{p}',s') = \\
= \;& \left|\langle 0|\hat{b}(\vec{p},s)\hat{b}^\dagger(\vec{p}',s')|0\rangle = \langle 0|\{\hat{b}(\vec{p},s),\hat{b}^\dagger(\vec{p}',s')\}|0\rangle - \underbrace{\langle 0|\hat{b}^\dagger\hat{b}|0\rangle}_{=0}\right| = \\
= \;& \Theta(x^0-y^0)\sum_{s}\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p}\exp(-ip(x-y))(u(\vec{p},s)\overline{u}(\vec{p},s))_{jl} - \\
- \;& \Theta(y^0-x^0)\sum_{s}\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p}\exp(ip(x-y))(v(\vec{p},s)\overline{v}(\vec{p},s))_{jl} = \\
= \;& \int\frac{d^3\vec{p}}{(2\pi)^3}\left(\sum_{r=1,2}(\psi_r(\vec{p},x)\overline{\psi}_r(\vec{p},y))_{jl}\Theta(x^0-y^0) - \sum_{r=3,4}(\psi_r(\vec{p},x)\overline{\psi}_r(\vec{p},y))_{jl}\Theta(y^0-x^0)\right) = \\
= \;& [iS_F(x-y)]_{jl}\,.
\end{aligned}
$$

Thus we see that this definition gives us the correct Green's function.

By doing the same we can also obtain that

$$
\{\hat{\psi}_j(\vec{x},t), \hat{\psi}_l^\dagger(\vec{y},t)\} = \ldots = \delta_{jl}\delta^3(\vec{x}-\vec{y}).
$$

This is called *micro causality.*

Remark From the formula we can see that this contains the Pauli principle of exclusion, because in the limit $t \to 0$ we see that

$$x^0 > y^0 \to \hat{\psi}_j(x)\hat{\overline{\psi}}_l(\vec{y}, x^0), \quad y^0 > x^0 \to -\hat{\overline{\psi}}_l(\vec{y}, x^0)\hat{\psi}_j(x).$$

Thus commutation gives us a minus sign. We cannot bring two particles in one state.

Now we are going to ask if we can apply this formalism to more than two *field operators.* In order to answer this we have to ask what we are going to describe in the field theory. We want to describe the quantum mechanical amplitudes for any process. This can be done by Feynman diagrams. Examples are shown in figure 4.1.

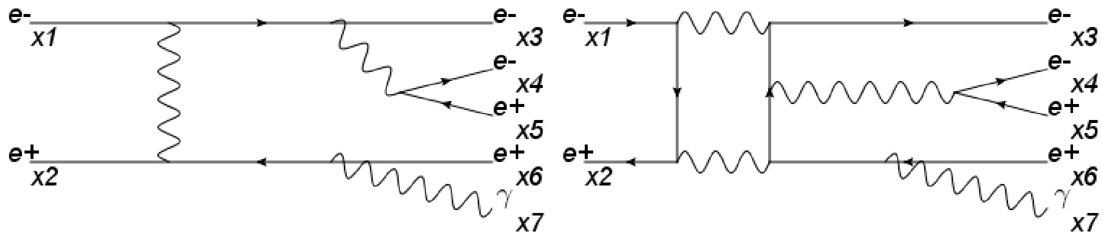

Figure 4.1: Two Feynman diagrams with the same initial and final states but complete different processes

We require an equation which gives us all possible amplitudes coherently. We know that $\hat{\psi}^\dagger(x)$ creates an electron of some impuls at location x or annihilates a positron. But $\hat{\psi}^\dagger(x)$ just contains the free plain waves in the sense of $\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_I(x)$ with

$$\mathcal{L}_0(x) = \overline{\psi}(x)(\hat{\not{p}} - m)\psi(x).$$

We only know the $\hat{\psi}_I$ which belong to the solution of the free Dirac equation, but we do need the $\hat{\psi}_H$ for the whole interaction, i.e.

$$\begin{aligned} & \langle 0|T\left\{\hat{\overline{\psi}}_H(x_1)\hat{\overline{\psi}}_H(x_2)\hat{\psi}_H(x_3)\hat{\psi}_H(x_4)\hat{\overline{\psi}}_H(x_5)\hat{\overline{\psi}}_H(x_6)\hat{A}^\mu_H(x_7)\right\}|0\rangle = \\ = \ & \langle 0|T\left\{\hat{\overline{\psi}}_I(x_1)\cdots\hat{A}^\mu_I(x_7)\exp\left(-i\int_{-\infty}^{\infty}\hat{\mathcal{H}}_I(x)d^4x\right)\right\}|0\rangle, \end{aligned}$$

with $\hat{\mathcal{H}}_I(x) = -\mathcal{L}_I(x)$. The index H symbolizes the Heisenberg picture. We picked the Heisenberg picture, because $|0\rangle$ is time-independent in this formalism. For futher calculations we have to transform the given equation into the Interaction picture with

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I.$$

For the transformation we have to remember the different pictures (Schrödinger, Heisenberg and interaction). Physically only the matrix elements are relevent, i.e.

$$\langle \Phi' | \hat{O} | \Phi \rangle \equiv \hat{O}_{\Phi'\Phi}.$$

This brings us to

$$\frac{d}{dt}\hat{O}_{\Phi'\Phi} = \left(\frac{d}{dt}\langle \Phi'|\right)\hat{O}|\Phi\rangle + \langle \Phi'|\left(\frac{d}{dt}\hat{O}\right)|\Phi\rangle + \langle \Phi'|\hat{O}\left(\frac{d}{dt}|\Phi\rangle\right).$$

4.1 The Schrödinger, Heisenberg and Interaction picture

We will now investigate the main differences between the various pictures. We use that

$$\begin{aligned} H(t) &= \int d^3x \mathcal{H}(x), \\ L(t) &= \int d^3x \mathcal{L}(x), \\ S &= \int dt L = \int d^4x \mathcal{L}(x). \end{aligned}$$

- In the Schrödinger picture we have

$$\begin{aligned} i\frac{d}{dt}|\Phi_S(t)\rangle &= \hat{H}|\Phi_S(t)\rangle, \\ \frac{d}{dt}\hat{O}_S &= 0, \\ -i\frac{d}{dt}\langle \Phi_S(t)| &= \langle \Phi_S(t)|\hat{H}. \end{aligned}$$

  Therefore we obtain

$$-i\frac{d}{dt}\hat{O}_{\Phi'\Phi} = \langle \Phi'_S(t)|[\hat{H}, \hat{O}_S]|\Phi_S(t)\rangle.$$

- In the Heisenberg picture we have

$$\begin{aligned}\frac{d}{dt}|\Phi_H(t)\rangle &= 0,\\ -i\frac{d}{dt}\hat{O}_H &= [\hat{H},\hat{O}_H],\\ \frac{d}{dt}\langle\Phi_H(t)| &= 0.\end{aligned}$$

Therefore we obtain

$$-i\frac{d}{dt}\hat{O}_{\Phi'\Phi} = \langle\Phi'_H|[\hat{H},\hat{O}_H]|\Phi_H\rangle.$$

- In the Interaction picture we have

$$\begin{aligned}i\frac{d}{dt}|\Phi_I(t)\rangle &= \hat{H}_I|\Phi_I(t)\rangle,\\ -i\frac{d}{dt}\hat{O}_I &= [\hat{H}_0,\hat{O}_I],\\ -i\frac{d}{dt}\langle\Phi_I(t)| &= \langle\Phi_I(t)|\hat{H}_I.\end{aligned}$$

Therefore we obtain

$$-i\frac{d}{dt}\hat{O}_{\Phi'\Phi} = \langle\Phi'_I(t)|[\hat{H},\hat{O}_I]|\Phi_I(t)\rangle.$$

We can solve the differential equation of the Interaction picture. We see that

$$\begin{aligned}|\Phi_I(t)\rangle &= T\left\{\exp\left(-i\int_{t_{-\infty}}^{t}\hat{H}_I(\tau)d\tau\right)\right\}|\Phi_I(t_{-\infty})\rangle,\\ i\frac{d}{dt}|\Phi_I(t)\rangle &= T\left\{-i\hat{H}_I(t)\exp\left(-i\int_{t_{-\infty}}^{t}\hat{H}_I(\tau)d\tau\right)\right\}|\Phi_I(t_{-\infty})\rangle = \hat{H}_I(t)|\Phi_I(t)\rangle.\end{aligned}$$

For the relation between the Schrödinger and the Interaction picture we get

$$\hat{O}_I = \exp(i\hat{H}_0(t-t_{-\infty})\hat{O}_S\exp(-i\hat{H}_0(t-t_{-\infty})).$$

Analog we get the relation between the Heisenberg and the Interaction picture,

$$\hat{O}_H = \left[T\left\{ \exp\left(-i \int_{t_{-\infty}}^{t} d\tau \hat{H}_I(\tau) \right) \right\} \right]^\dagger \hat{O}_I \left[T\left\{ \exp\left(-i \int_{t_{-\infty}}^{t} d\tau' \hat{H}_I(\tau') \right) \right\} \right].$$

With $|0\rangle_H = |0\rangle_{t_{-\infty}} = 0\rangle_{I,t_{-t\infty}}$ and the equations above we calculate

$$|0\rangle_I = T\left\{ \exp\left(-i \int_{t_{-\infty}}^{t} d\tau \hat{H}_I(\tau) \right) \right\} |0\rangle_H.$$

Therefore we calculate with

$$0\rangle_{H,t_{+\infty}} = |0\rangle_{I,t_{+\infty}} = T\left\{ \exp\left(-i \int_{t_{-\infty}}^{t_{+\infty}} d\tau \hat{H}_I(\tau) \right) \right\} |0\rangle_H,$$

that, with the assumption that without loss of generality we have $t_1 > t_2 > \cdots t_N$, we find

$$\begin{aligned}
& T\left\{ \exp\left(-i \int_{t_{-\infty}}^{t_{+\infty}} d\tau \hat{H}_I(\tau) \right) \right\} \left[T\left\{ \exp\left(-i \int_{t_{-\infty}}^{t_1} d\tau \hat{H}_I(\tau) \right) \right\} \right]^\dagger = \\
= \;& T\left\{ \exp\left(-i \int_{t_1}^{t_{+\infty}} \cdots \right) \right\} T\left\{ \exp\left(-i \int_{t_{-\infty}}^{t_1} \cdots \right) \right\} \left[T\left\{ \exp\left(-i \int_{t_{-\infty}}^{t_1} \cdots \right) \right\} \right]^\dagger = \\
= \;& T\left\{ \exp\left(-i \int_{t_1}^{t_{+\infty}} d\tau \hat{H}_I(\tau) \right) \right\}, \\
\Rightarrow \;& \langle 0| T\left\{ \exp\left(-i \int_{t_1}^{t_{+\infty}} d\tau \hat{H}_I(\tau) \right) \right\} \hat{\Phi}_I(y_1) T\left\{ \exp\left(-i \int_{t_2}^{t_1} d\tau \hat{H}_I(\tau) \right) \right\} \hat{\Phi}_I(y_2) \cdots \\
& \cdots T\left\{ \exp\left(-i \int_{t_N}^{t_{N-1}} d\tau \hat{H}_I(\tau) \right) \right\} \hat{\Phi}_I(y_N) T\left\{ \exp\left(-i \int_{t_{-\infty}}^{t_N} d\tau \hat{H}_I(\tau) \right) \right\} |0\rangle = \\
= \;& \langle 0| T\left\{ \hat{\Phi}_I(y_1) \cdots \hat{\Phi}_I(y_N) \exp\left(-i \int_{t_{-\infty}}^{t_{+\infty}} d\tau \hat{H}_I(\tau) \right) \right\} |0\rangle.
\end{aligned}$$

Finally we found an expression for any amplitude possible in quantum field theory. The rest is just mathematics.

4.2 Wick's theorem

Theorem For $2N$ (even) field operators we get

$$\begin{aligned}
&\langle 0|T\left\{\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_{2N})\right\}|0\rangle = \\
= \quad &\langle 0|T\left\{\hat{\Phi}(x_1)\hat{\Phi}(x_2)\right\}|0\rangle\langle 0|T\left\{\hat{\Phi}(x_3)\hat{\Phi}(x_4)\right\}|0\rangle\cdots\langle 0|T\left\{\hat{\Phi}(x_{2N-1})\hat{\Phi}(x_{2N})\right\}|0\rangle + ...
\end{aligned}$$

plus permutations of this expression. We see that

$$\langle 0|T\left\{\hat{\Phi}(x_1)\hat{\Phi}(x_2)\right\}|0\rangle = iD_F(x-y)$$

is our propagator. Thus we have N propagators. For $2N+1$ (odd) field operators we get 0 as result.

Proof In order to proof this we introduce the normal ordered product

$$:\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N): \quad ,$$

which gives us alignment where all creation operators are placed before the annihilation operators. We will then proof the general version with

$$\begin{aligned}
&T\left\{\hat{\Phi}(x_1)\cdots\hat{\Phi}(x_{2N})\right\} =: \hat{\Phi}(x_1)\cdots\hat{\Phi}(x_N): + \sigma\left(\langle 0|T\left\{\hat{\Phi}(x_1)\hat{\Phi}(x_2)\right\}|0\rangle : \hat{\Phi}(x_3)\cdots\hat{\Phi}(x_N)\right) \\
+ \quad &\sigma\left(\langle 0|T\left\{\hat{\Phi}(x_1)\hat{\Phi}(x_2)\right\}|0\rangle\langle 0|T\left\{\hat{\Phi}(x_3)\hat{\Phi}(x_4)\right\}|0\rangle : \hat{\Phi}(x_5)\cdots\hat{\Phi}(x_N)\right) + ...,
\end{aligned}$$

where the $\sigma(a)$ means the sum of all permutations of a. Because of

$$\langle 0| :\cdots: |0\rangle = 0$$

we will see that the main idea for this proof will be

$$T\left\{\hat{\Phi}(x_1)\hat{\Phi}(x_2)\right\} - :\hat{\Phi}(x_1)\hat{\Phi}(x_2): = f(x) \neq \hat{O}.$$

Since the outcome is a function (distribution to be more specific) and not an operator we will see that this will lead to a proof, which can be done by mathematical induction. There

$$N = 1, \quad T\{\hat{\Phi}(x_1)\} =: \hat{\Phi}(x_1):,$$

is trivial. For the $N \to N+1$ step we say that without loss of generality we can say that

$$t_{N+1} < t_i, \qquad \forall i \in \{1, ..., N\}.$$

So we get

$$\begin{aligned} & T\{\hat{\Phi}(x_1) \cdots \hat{\Phi}(x_{N+1})\} = T\{\hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N)\}\hat{\Phi}(x_{N+1}) = \\ = \; & : \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_N) : \hat{\Phi}(x_{N+1}) + \sigma(\langle 0|T\{\hat{\Phi}(x_1)\hat{\Phi}(x_2)\}|0\rangle : \hat{\Phi}(x_3) \cdots \hat{\Phi}(x_N) : \hat{\Phi}(x_{N+1})) + ... \end{aligned}$$

We obtained this with our induction condition. We now divide $\hat{\Phi}(x_{N+1})$ in a part with the creation operator $\hat{\Phi}^e$ and a part with the annihilation operator $\hat{\Phi}^v$. Non-trivial is only the commutation of $\hat{\Phi}^e(x_{N+1})$ with the normal ordered products. We will do this in the next calculation,

$$: \hat{\Phi}(x_k) \cdots \hat{\Phi}(x_N) : \hat{\Phi}^e(x_{N+1}).$$

To solve this we introduce the index set of the creation parts, $E = \{k, k+1, ..., N\}$ and the index set of the annihilation parts, $V = \{k, k+1, ..., N\}$. We see that

$$E \cup V = \{k, ..., N\}, \qquad E \cap V = \emptyset.$$

Thus we calculate that

$$\begin{aligned} & \sum_{E,V} \left(\prod_{i \in E} \hat{\Phi}^e(x_i) \right) \left(\prod_{j \in V} \hat{\Phi}^v(x_j) \right) \hat{\Phi}^e(x_{N+1}) = \\ = \; & \sum_k \sum_{E,V} \left(\prod_{i \in E} \hat{\Phi}^e(x_i) \right) \left(\prod_{j \in V, j \neq k} \hat{\Phi}^v(x_j) \right) \left[\hat{\Phi}^v(x_k), \hat{\Phi}^e(x_{N+1}) \right] + ... \end{aligned}$$

Since the commutator is a function $f(x_k, x_{N+1})$ we obtain by viewing the vacuum expectation value (VEV),

$$f(x_k, x_{N+1})\langle 0|0\rangle = \langle 0| \left[\hat{\Phi}^v(x_k), \hat{\Phi}^e(x_{N+1}) \right] |0\rangle = \langle 0|T\{\hat{\Phi}^v(x_k)\hat{\Phi}^e(x_{N+1})\}|0\rangle.$$

For the missing part we get

$$... = \sum_{E,V} \left(\prod_{i \in \{E,N+1\}} \hat{\Phi}^e(x_i) \right) \left(\prod_{j \in V} \hat{\Phi}^v(x_j) \right).$$

This is equivalent to all new terms for $N+1$.

5 The Feynman rules

5.1 Electron-Electron-Photon-Vertex

The Feynman diagram for such a process is shown in figure (5.1).

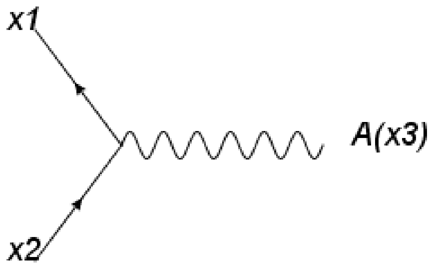

Figure 5.1: Electron-Electron-Photon-Vertex

The amplitude can be written as

$$\langle 0|T\{\hat{\psi}_{j_1}(x_1)\hat{\bar{\psi}}_{j_2}\hat{A}_\nu(x_3)\exp\left(-i\int_{-\infty}^{\infty} d^4y\hat{\bar{\psi}}(y)e\hat{\not{A}}(y)\hat{\psi}(y)\right)\}|0\rangle.$$

We now only consider the linear term of the taylor series expansion of the exponential function. Therefore we get

$$\begin{aligned}
& \int d^4y\langle 0|T\{\hat{\psi}_{j_1}(x_1)\hat{\bar{\psi}}_{j_2}(x_2)\hat{A}_\nu(x_3)(-ie\gamma_\mu)_{k_2k_1}\hat{\bar{\psi}}_{k_2}(y)\hat{\psi}_{k_1}(y)A^\mu(y)\}|0\rangle = \\
= \quad & \int d^4y\langle 0|T\{\hat{A}_\nu(x_3)\hat{A}^\mu(y)\}|0\rangle(-ie\gamma_\mu)_{k_2k_1}\langle 0|T\{\hat{\psi}_{j_1}(x_1)\hat{\bar{\psi}}_{j_2}(x_2)\hat{\bar{\psi}}_{k_2}(y)\hat{\psi}_{k_1}(y)\}|0\rangle = \\
= \quad & \int d^4y\langle 0|T\{\hat{A}_\nu(x_3)\hat{A}^\mu(y)\}|0\rangle(-ie\gamma_\mu)_{k_2k_1}\cdot \\
\cdot \quad & (\langle 0|T\{\hat{\psi}_{j_1}(x_1)\hat{\bar{\psi}}_{j_2}(x_2)\}|0\rangle\langle 0|T\{\hat{\bar{\psi}}_{k_2}(y)\hat{\psi}_{k_1}(y)\}|0\rangle - \\
- \quad & \langle 0|T\{\hat{\psi}_{j_1}(x_1)\hat{\bar{\psi}}_{k_2}(y)\}|0\rangle\langle 0|T\{\hat{\bar{\psi}}_{j_2}(x_2)\hat{\psi}_{k_1}(y)\}|0\rangle).
\end{aligned}$$

The first term gives us figure (5.2). This only gives us a contribution for different photon momenta, which is something like $\delta(\omega)$ with the photon energy $\omega\hbar$.

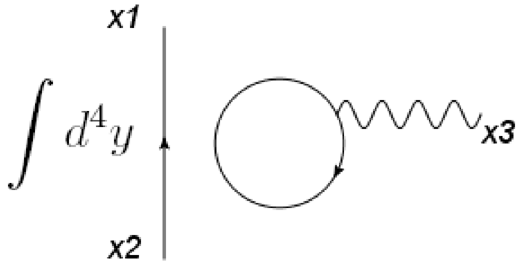

Figure 5.2: The first term of our amplitude calculation

For the second term we have figure(5.3).

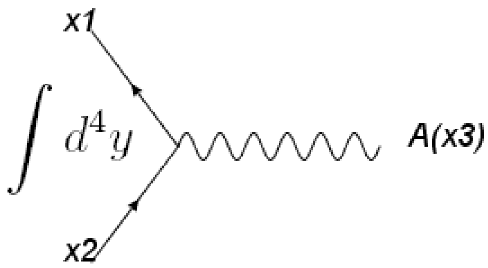

Figure 5.3: The second term of our amplitude calculation

With

$$(iS_F(x_1 - y))_{j_1k_2} = \langle 0|T\{\hat{\psi}_{j_1}(x_1)\hat{\overline{\psi}}_{k_2}(y)\}|0\rangle = -\langle 0|T\{\hat{\overline{\psi}}_{k_2}(y)\hat{\psi}_{j_1}(x_1)\}|0\rangle,$$

we get for the amplitude

$$\int d^4y iD_{F,\mu\nu}(x_3 - y)\underbrace{(iS_F(x_1 - y))_{j_1k_2}(-ie\gamma^\mu)_{k_2k_1}(iS_F(y - x_2))_{k_1j_2}}_{=(iS_F(x_1-y))(-ie\gamma^\mu)(iS_F(y-x_2)))_{j_1j_2}}.$$

In momentum space the whole equation becomes even more simple,

$$\begin{aligned}
S_F(x_1 - y) &= \int \frac{d^4p_1}{(2\pi)^4} \exp(-ip_1 \cdot (x_1 - y)) \frac{\not{p}_1 + m}{p_1^2 - m^2 + i\varepsilon}, \\
S_F(y - x_2) &= \int \frac{d^4p_2}{(2\pi)^4} \exp(-ip_2 \cdot (y - x_2)) \frac{\not{p}_2 + m}{p_2^2 - m^2 + i\varepsilon}, \\
D_{F,\mu\nu}(x_3 - y) &= -\int \frac{d^4p_3}{(2\pi)^4} \exp(-ip_3 \cdot (x_3 - y)) \frac{g_{\mu\nu}}{p_2^2 + i\varepsilon}.
\end{aligned}$$

For the last part we had to decide for a gauge rule. We picked the Feynman gauge. Inserting the momenta versions gives us

$$\begin{aligned}
&\int d^4y \int \frac{d^4p_1 d^4p_2 d^4p_3}{(2\pi)^{12}} \exp(-iy \cdot (p_2 - p_1 - p_3)) \exp(-ip_3x_3) \exp(ip_2x_2) \exp(-ip_1x_1) \cdot \\
&\cdot \left(-i\frac{g_{\mu\nu}}{p_3^2 + i\varepsilon}\right) \left(i\frac{\not{p}_1 + m}{p_1^2 - m^2 + i\varepsilon}\right) (-ie\gamma^\mu) \left(i\frac{\not{p}_2 + m}{p_2^2 - m^2 + i\varepsilon}\right).
\end{aligned}$$

We directly see that the integration over $\int d^4y$ gives us $(2\pi)^4\delta^4(p_2 - p_1 - p_3)$. Therefore we see that $p_3 = p_2 - p_1$ which is the conservation of energy and momentum. We can then use this to obtain

$$\begin{aligned}
&\int d^4p_1 \int d^4p_2 \frac{1}{(2\pi)^8} \exp(ip_2 \cdot (x_2 - x_3)) \exp(-ip_1 \cdot (x_1 - x_3)) \cdot \\
&\cdot \left(-i\frac{g_{\mu\nu}}{(p_2 - p_1)^2 + i\varepsilon}\right) \left(i\frac{\not{p}_1 + m}{p_1^2 - m^2 + i\varepsilon}\right) (-ie\gamma^\mu) \left(i\frac{\not{p}_2 + m}{p_2^2 - m^2 + i\varepsilon}\right).
\end{aligned}$$

This is what we expected combined with a new term which describes the 'Electron-Photon-Vertex'. This is a new feynman rule.

5.2 The quantization of the photon field

The photon field has only 2 degrees of freedom. This can be seen from the Planck equation for spectral energy density,

$$u(\omega) = \frac{N}{2} \frac{\hbar}{c^3\pi^2} \frac{\omega^3}{\exp(\hbar\omega/k_BT) - 1},$$

where N are the degrees of freedom. For a photon field we find $N = 2$. We also see that plain waves are transversal polarized. In order to have everything Lorentz invariant, i.e. A_μ with $\mu = 0, 1, 2, 3$, we have to think a bit. The Lagrange density for electromagnetic fields is

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x).$$

Now get our equation of motion,

$$\partial^\sigma F_{\sigma\lambda}(x) = 0.$$

This can be written as

$$\partial^\sigma \partial_\sigma A_\lambda(x) - \partial^\sigma \partial_\lambda A_\sigma(x) = 0. \tag{5.1}$$

By inserting the Fourier transformation of

$$A^\mu(x) = \int \frac{d^4 q}{(2\pi)^4} \exp(-iq \cdot x) A^\mu(q),$$

in equation 5.1 we obtain that

$$q_\mu q^\mu A^\nu(q) - q_\mu q^\nu A^\mu(q) = 0. \tag{5.2}$$

Without loss of generality we can choose the z direction as the momentum propagation, i.e. $q_1 = q_2 = 0$.

- *1st case* We have $q^2 \neq 0$. We get that $\nu = 0, 3$ are always fulfilled. For $\nu = 1, 2$ we see that $A^1(x) = A^2(x) = 0$ is required.
- *1st subcase* We have $q^2 > 0$ with $q_0 \neq 0$. So we see

  $$((q^0)^2 - (q^3)^2) A^3 - q^3 (q^0 A^0 - q^3 A^3) = 0 \quad \Rightarrow \quad (q^0)^2 A^3 - q^3 q^0 A^0 = 0.$$

  Therefore we get the relation

  $$\frac{A^3}{A^0} = \frac{q^3}{q^0} \qquad \rightsquigarrow \qquad A_\mu \propto q_\mu.$$

- *2nd subcase* We have $q^2 < 0$ with $q_3 \neq 0$. So we see

$$((q^0)^2 - (q^3)^2)A^0 - q^0(q^0A^0 - q^3A^3) = 0 \quad \Rightarrow \quad q^3A^0 - q^0A^3 = 0.$$

Therefore we get the same relation

$$\frac{A^3}{A^0} = \frac{q^3}{q^0} \qquad \rightsquigarrow \qquad A_\mu \propto q_\mu.$$

- *2nd case* We have $q^2 = 0$. We obtain

$$q_\mu A^\mu = 0 \quad \Rightarrow \quad q_0A^0 - q_3A^3 = 0.$$

So we have the relation

$$\frac{A^3}{A^0} = \frac{q^0}{q^3} \stackrel{!}{=} \frac{q^3}{q^0}.$$

Overall we get that

$$A^\mu(q) = \begin{pmatrix} 0 \\ A^1(q) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ A^2(q) \\ 0 \end{pmatrix} + a(q)q^\mu, \qquad \forall q_\lambda. \tag{5.3}$$

Every photon couples somewhere on a conserved charge. An illustration can be found in 5.4.

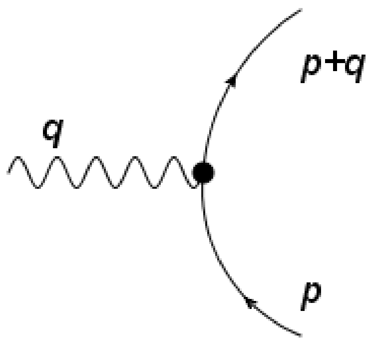

Figure 5.4: Photon coupling on a conserved charge j^μ

The continuity equation says that

$$\partial_\mu j^\mu(x) = 0 \quad \Rightarrow \quad q_\mu j^\mu(q) = 0.$$

So we can also do canonical quantization with

$$\hat{A}^\mu(x) = \int \frac{d^3q}{(2\pi)^3 2q_0} \sum_{\lambda=1}^{2} \left[\epsilon^\mu(\vec{q},\lambda)\exp(-iq\cdot x)\hat{a}^\lambda(q) + (\epsilon^*)^\mu(\vec{q},\lambda)\exp(iq\cdot x)(\hat{a}^\lambda)^\dagger(q)\right],$$

with $(\epsilon^\mu(\vec{q},1)) = (0,1,0,0)$ and so on for $\vec{q} = q\vec{e}_3$. This is equivalent to

$$\hat{A}^\mu(x) = \int \frac{d^3q}{(2\pi)^3 2q_0} \sum_{\lambda=0}^{3} [...] \,.$$

We can use this. With the definition that

$$\sum_\lambda \epsilon^\mu(\vec{q},\lambda)(\epsilon^*)^\nu(\vec{q},\lambda) = g^{\mu\nu}$$

it is possible to make a quantization with

$$\left[\hat{a}(\vec{q},\lambda), \hat{a}^\dagger(\vec{q}',\lambda')\right] = -g^{\lambda\lambda'} 2q^0 (2\pi)^3 \delta^3(\vec{q}-\vec{q}').$$

Now we rewrite equation 5.2 to obtain

$$[q_\mu q^\mu g^\nu{}_\lambda - q_\lambda q^\nu] A^\lambda = 0.$$

We are interested in the propator. Therefore we insert the propagator and see

$$[q_\mu q^\mu g^\nu{}_\lambda - q_\lambda q^\nu](D_F(q))^\lambda{}_\sigma = -g^\nu{}_\sigma + b\frac{q^\nu q_\sigma}{q^2}.$$

The b is an arbitrary constant. This is possible since the contribution of this term on

$$A^\lambda(x) = \int d^4y (D_F(x-y))^\lambda{}_\sigma j^\sigma(y)$$

is zero. So we can make an ansatz for D_F now,

$$(D_F(q))^{\lambda}{}_{\sigma} = g^{\lambda}{}_{\sigma} B(q^2) + \frac{q^{\lambda} q_{\sigma}}{q^2} A(q^2).$$

The unknown functions A and B can only have an argument of q^2 since q^2 is a Lorentz-scalar. Inserting this ansatz gives us

$$q^2 g^{\nu}{}_{\sigma} B(q^2) - q^{\nu} q_{\sigma} B(q^2) + q^{\nu} q_{\sigma} A(q^2) - q^{\nu} q_{\sigma} A(q^2) \stackrel{!}{=} -g^{\nu}{}_{\sigma} + b\frac{q^{\nu} q_{\sigma}}{q^2}.$$

We see directly that this is solved for any $A(q^2)$ with $B = -1/q^2$ and $b = 1$. Therefore we found that

$$[D_F(q)]^{\lambda}{}_{\sigma} = -\frac{g^{\lambda}{}_{\sigma} - q^{\lambda} q_{\sigma} A(q^2)}{q^2}. \tag{5.4}$$

The freedom of choice for $A(q^2)$ is equivalent to the gauge freedom. The most famous choices are,

- the *Feynman gauge*. Here we have $A = 0$, which gives us

$$(D_F(q))^{\lambda}{}_{\sigma} = -\frac{g^{\lambda}{}_{\sigma}}{q^2 + i\varepsilon}.$$

- the *Landau gauge*. Here we have $A = -1/q^2$, which gives us

$$(D_F(q))^{\lambda}{}_{\sigma} = -\frac{g^{\lambda}{}_{\sigma} - \frac{q^{\lambda} q_{\sigma}}{q^2}}{q^2 + i\varepsilon}.$$

  This choice can be useful for very complex structures with lots of propagators.

5.3 Summary

We will now summarize the Feynman rules of quantum electrodynamics.

- For an incoming particle we have to write $u(p,s)$. We will not write the additional $1/\sqrt{2E}\exp(-ipx)$ factor (but we should not forget about this factor).
- For an outgoing particle we have to write $\overline{u}(p,s)$. The additional factor is $1/\sqrt{2E}\exp(ipx)$.
- For an incoming anti-particle we have to write $\overline{v}(p,s)$. The additional factor is $1/\sqrt{2E}\exp(-ipx)$.

- For an outgoing anti-particle we have to write $v(p,s)$. The additional factor is $1/\sqrt{2E}\exp(ipx)$.
- For an electron-photon vertex we have to write $(-ie\gamma_\mu)$. The additional factor of $(2\pi)^4\delta^4(p-p'-q)$ can be left out as well.
- For a photon in the initial state we have to write ϵ^μ. The additional factor is $1/\sqrt{2E}\exp(-ipx)$.
- For a photon in the final state we have to write $(\epsilon^\mu)^*$. The additional factor is $1/\sqrt{2E}\exp(ipx)$.
- The Dirac-propagator is given by

$$i\frac{\not{p}+m}{p^2-m^2+i\varepsilon}.$$

- The photon propagator with an arbitrary c (can be set 1 in quantum electrodynamics) is given by

$$-i\frac{g_{\mu\nu}-\frac{1-c}{c}\frac{q_\mu q_\nu}{q^2}}{q^2+i\varepsilon}.$$

- Loops are a bit more complicated and will be discussed later on (about renormalization). The commutation of fermion lines gives us always a (-1) factor.

6 Calculating physical processes

6.1 Electron-Myon-Scattering

We will now take a look at Electron-Myon-Scattering. A myon is something like a big electron. If you look at the Standard model you will see three families,

$$\left[\begin{pmatrix} \nu_e \\ e \end{pmatrix}\begin{pmatrix} u \\ d \end{pmatrix}_{3c}\right], \quad \left[\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}\begin{pmatrix} c \\ s \end{pmatrix}_{3c}\right], \quad \left[\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}\begin{pmatrix} t \\ b \end{pmatrix}_{3c}\right],$$

with the muon in the second family. The process is shown in figure 6.1.

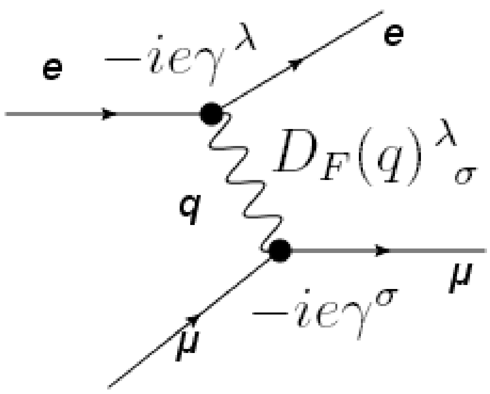

Figure 6.1: Second order process (for the series expansion of the exponential function)

An electron with s_e, p_e exchanges momentum with an myon of s_μ, p_μ. Therefore we get $s_e, p_e \to s'_e, p'_e$, $s_\mu, p_\mu \to s'_\mu, p'_\mu$ and for the virtual photon $q = p_e - p'_e = p'_\mu - p_\mu$. Our photon propagator in momentum space contained all possible polarizations and states, our electron-photon-vertex is

$$-ie\gamma_\varrho \left((2\pi)^4 \delta^4(p_e - p'_e - q)\right),$$

with the energy-momentum conservation term in shape of the δ-distribution. We will use these terms implicitly to shorten the calculations. For the myon-photon-vertex we have

$$-ie\gamma_\nu \left((2\pi)^4\delta^4(p_\mu - p'_\mu + q)\right).$$

The incoming and outgoing plane-wave equations are

$$\begin{array}{llll} u_e(p_e,s_e) & \frac{1}{\sqrt{2E_e}} & [\exp(-ip_e x)]\,, & \overline{u}_e(p'_e,s'_e)\dfrac{1}{\sqrt{2E'_e}}\,[\exp(ip'_e x)]\,, \\ u_\mu(p_\mu,s_\mu) & \frac{1}{\sqrt{2E_\mu}} & [\cdots]\,, & \overline{u}_\mu(p'_\mu,s'_\mu)\dfrac{1}{\sqrt{2E'_\mu}}\,[\cdots]\,. \end{array}$$

The exponential functions become δ-distributions due to integration over them. Therefore we won't write them explicitly in order to shorten the calculations. The quantum mechanical ampltiude is

$$\begin{aligned} M &= \frac{-1}{\sqrt{16E_eE'_eE_\mu E'_\mu}}e^2\overline{u}(p'_e,s'_e)\gamma_\varrho u(p_e,s_e)\overline{u}(p'_\mu,s'_\mu)\gamma_\nu u(p_\mu,s_\mu)\,\cdot \\ &\quad \cdot\frac{-g^{\varrho\nu}}{(p_e-p'_e)^2+i\varepsilon}(2\pi)^4\delta^4(p_e+p_\mu-p'_e-p'_\mu). \end{aligned}$$

We will now calculate the probability density, which can be retrieved by calculating $|M|^2$. We perform

$$\begin{aligned} |M|^2 &= e^4\left[\overline{u}_e(p'_e,s'_e)\gamma_\varrho u_e(p_e,s_e)\right]^\dagger \overline{u}_e(p'_e,s'_e)\gamma_{\varrho'}u_e(p_e,s_e)\,\cdot \\ &\quad \cdot\left[\overline{u}_\mu(p'_\mu,s'_\mu)\gamma_\nu u_\mu(p_\mu,s_\mu)\right]^\dagger \overline{u}_\mu(p'_\mu,s'_\mu)\gamma_{\nu'}u_\mu(p_\mu,s_\mu)\,\cdot \\ &\quad \cdot\frac{g^{\mu\nu}g^{\mu'\nu'}}{((p_e-p'_e)^2+i\varepsilon)^2}\frac{1}{16E_eE'_eE_\mu E'_\mu}\left((2\pi)^4\delta^4(p_e+p_\mu-p'_e-p'_\mu)\right)^2. \end{aligned}$$

We have to take a closer look at this - explicitly at the $\delta(\cdot)\delta(\cdot)$ term. This is not well defined. Now we look at the conjugate transpose with $\gamma_\varrho^\dagger = \gamma_0\gamma_\varrho\gamma_0$. The electric currents are hermitian. We see

$$\left[\overline{u}_e(p'_e,s'_e)\gamma_\varrho u_e(p_e,s_e)\right]^\dagger = u_e^\dagger(p_e,s_e)\gamma_0\gamma_\varrho\gamma_0\gamma_0 u_e(p'_e,s'_e) = \overline{u}_e(p_e,s_e)\gamma_\varrho u_e(p'_e,s'_e).$$

The first row of $|M|^2$ containts a term which can be written like the trace of it, because we use that

$$\vec{a}^T\vec{b} = \sum_i a_i b_i = \mathrm{tr}\{\vec{b}\vec{a}^T\}.$$

Through extensive usage of our projection operators (defined as e.g. $u_e(p'_e, s'_e)\overline{u}_e(p'_e, s'_e)$) we see that

$$\begin{aligned} |M|^2 &\propto \mathrm{tr}\left\{\gamma_\varrho(\not{p}'_e + m)\frac{1+\gamma_5\not{s}'_e}{2}\gamma_{\varrho'}(\not{p}_e + m)\frac{1+\gamma_5\not{s}_e}{2}\right\} = \\ &= \frac{1}{4}\mathrm{tr}\left\{\gamma_\varrho(\not{p}'_e + m)(1+\gamma_5\not{s}'_e)\gamma_{\varrho'}(\not{p}_e + m)(1+\gamma_5\not{s}_e)\right\}. \end{aligned}$$

The advantage of our usage of the trace operation is that we are allowed to commutate the elements of the trace. Since we are not interested in the spin components we have to sum over all outcoming spins and make the average of all incoming spins with $\frac{1}{2}\frac{1}{2}\sum_{s_e,s_\mu}\sum_{s'_e,s'_\mu}|M|^2$. Overall we have

$$\begin{aligned} \frac{1}{4}\sum_{s_e,s'_e,s_\mu,s'_\mu}|M|^2 &= \frac{1}{4}e^4\frac{1}{4}\mathrm{tr}\left\{\gamma_\varrho(\not{p}'_e + m_e)2\gamma_{\varrho'}(\not{p}_e + m_e)2\right\}\cdot \\ &\cdot\mathrm{tr}\left\{\gamma_\nu(\not{p}'_\mu + m_\mu)\gamma_{\nu'}(\not{p}_\mu + m_\mu)\right\}\cdot \\ &\cdot\frac{g^{\varrho\nu}g^{\varrho'\nu'}}{((p_e - p'_e)^2 + i\varepsilon)^2}\frac{1}{16E_eE'_eE_\mu E'_\mu}\left((2\pi)^4\delta^4(p_e + p_\mu - p'_e - p'_\mu)\right)^2. \end{aligned}$$

The traces can be calculated with the identities of the γ matrices. We know that

$$\mathrm{tr}\left\{\gamma_\varrho(\not{p}'_e + m_e)\gamma_{\varrho'}(\not{p}_e + m_e)\right\} = \underbrace{\mathrm{tr}\{\gamma_\varrho\not{p}'_e\gamma_{\varrho'}\not{p}_e\}}_{=4((p'_e)_\varrho(p_e)_{\varrho'} - g_{\varrho\varrho'}p'_ep_e + (p_e)_\varrho(p'_e)_{\varrho'})} + \underbrace{\mathrm{tr}\{\gamma_\varrho\gamma_{\varrho'}\}}_{=4g_{\varrho\varrho'}}m_e^2.$$

The mixed terms are all zero since there is an odd number of γ matrices under the trace-function. Overall we have

$$\mathrm{tr}\{\cdots\} = 4\left((p'_e)_\varrho(p_e)_{\varrho'} + (p'_e)_{\varrho'}(p_e)_\varrho + g_{\varrho\varrho'}(m_e^2 - p_ep'_e)\right).$$

To get a work around for our δ^2 problem we introduce a finite spacetime volume, L^3T. So we get

$$(2\pi)^4\delta^4(p) = \int d^4x \exp(-ipx) \quad \Rightarrow \quad (2\pi)^4\delta^4(0) = \int_{VT} d^4x = VT.$$

Now we need a new normalization factor. Our choice is

$$\psi \to \frac{1}{\sqrt{V}}\psi, \qquad \Rightarrow \frac{d}{dV}\left(\int_V d^3x\overline{\psi}\psi\right) = 0.$$

This new normalization would also change the dimension of $S_F(x-x')$. Since this is not intended we define

$$\int \frac{d^3p}{(2\pi)^3} \to V\int \frac{d^3p}{(2\pi)^3}.$$

This is equivalent to the sum over j_1, j_2, j_3 with $\vec{p} = \left(\frac{2\pi}{L}j_1, \frac{2\pi}{L}j_2, \frac{2\pi}{L}j_3\right)$. Now we only have to define an object, that does not depend on V and T. This is the cross section σ. The question is: With what probability hits the electron in a time T the surface A. When $\vec{v}$ is the velocity in the direction of the surface, we can say that this is given by

$$\frac{|\vec{v}|TA}{V}.$$

Definition The probability that a reaction is the consequence of the electron hitting the surface A is given by σ/A. The probability that a reaction follows is

$$\frac{vTA}{V}\frac{\sigma}{A} = \sigma\frac{vT}{V} = V^2\int_\Omega \frac{d^3p'_e d^3p'_\mu}{(2\pi)^3(2\pi)^3}\cdots|M|^2.$$

Since $|M|^2 \propto VT\frac{1}{V^4}$ we get that σ does not depend on V and T. So we can write that

$$\sigma = \int \frac{d^3p'_e d^3p'_\mu}{(2\pi)^6}\frac{1}{v}\frac{1}{4}\sum_{s_e,s'_e,s_\mu,s'_\mu}|\mathcal{M}|^2,$$

with $\mathcal{M}$ without V and T. Or more explicit

$$\begin{aligned}\sigma &= \int \frac{d^3p'_e d^3p'_\mu}{(2\pi)^2}\frac{\delta^4(p_e+p_\mu-p'_e-p'_\mu)}{vE_eE'_eE_\mu E'_\mu}\frac{e^4}{2}\cdot \\ &\quad\cdot\left[p'_e\cdot p'_\mu p_e\cdot p_\mu + p'_e\cdot p_\mu p_e\cdot p'_\mu - m_\mu^2 p'_e\cdot p_e - m_e^2 p'_\mu\cdot p_\mu + 2m_e^2m_\mu^2\right].\end{aligned}$$

We see directly that the term in $[\cdot]$ is Lorentz invariant. We are now going to see that the other term is also Lorentz invariant. Therefore we write

$$v = |\vec{v}| = |\vec{v}_e - \vec{v}_\mu| = \left| \frac{\vec{p}_e}{E_e} - \frac{\vec{p}_\mu}{E_\mu} \right| .$$

We analyse the case that $\vec{p}_e$ is in the opposite direction of $\vec{p}_\mu$. Therefore we go e.g. in the center of mass system (cms) or laboratory system,

$$|\vec{v}| = \frac{|\vec{p}_2|}{E_e} + \frac{|\vec{p}_\mu|}{E_\mu}, \quad \Rightarrow \quad E_e E_\mu |\vec{v}| = E_\mu |\vec{p}_e| + E_e |\vec{p}_\mu|.$$

On the other side we have

$$(p_e \cdot p_\mu)^2 - m_e^2 m_\mu^2 = (E_e E_\mu - \vec{p}_e \vec{p}_\mu)^2 - (E_e^2 - \vec{p}_e^2)(E_\mu^2 - \vec{p}_\mu^2).$$

Since the momenta are in opposite direction the angle between $\vec{p}_\mu$ and $\vec{p}_e$ is π. So we get

$$\begin{aligned} (p_e \cdot p_\mu)^2 - m_e^2 m_\mu^2 &= (E_e E_\mu + |\vec{p}_e||\vec{p}_\mu|)^2 - (E_e^2 - |\vec{p}_e|^2)(E_\mu^2 - |\vec{p}_\mu|^2) = \\ &= 2E_e E_\mu |\vec{p}_e||\vec{p}_\mu| + E_e^2 |\vec{p}_\mu|^2 + E_\mu^2 |\vec{p}_e|^2 = \left(|\vec{p}_e| E_\mu + E_e |\vec{p}_\mu| \right)^2 . \end{aligned}$$

Therefore $vE_e E_\mu = \sqrt{(p_e \cdot p_\mu)^2 - m_e^2 m_\mu^2}$ is Lorentz invariant. We also see that

$$\int \frac{d^3 p_e'}{(2\pi)^3} \frac{1}{E_e'} \cdots = \int \frac{d^4 p_e'}{(2\pi)^3} 2\delta(p_e'^2 - m^2) \cdots ,$$

so that the integral itself is Lorentz invariant. Overall we see that σ is invariant under Lorentz transformations.

For the experiment with particle scattering we do need the following components:

- An acceleration expert gives us the number of particles per area and second, which is called luminosity with unit $1/\mathrm{cm}^2/\mathrm{s}$,

$$L = \frac{N}{AT}.$$

- The theoretical physicist calculates the cross-section σ.
- The experimental physicist is interested in the count rate which can be calculated by

$$A = L \cdot \sigma.$$

A general form for a $2 \to n$ process looks like

$$d\sigma = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 ... - p_{n+2}) \cdot \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \underbrace{\frac{d^3 p_3}{(2\pi)^3 2E_3} \cdots \frac{d^3 p_{n+2}}{(2\pi)^3 2E_{n+2}}}_{\text{Lorentz invariant phase space}}.$$

In our case (Electron-Myon) we have

$$|\mathcal{M}|^2 = \left| e^2 \overline{u}_e(p_e', s_e') \gamma_\varrho \cdots u_\mu(p_\mu, s_\mu) \frac{-g^{\varrho\nu}}{(p_e - p_e')^2 + i\varepsilon} \right|^2 .$$

Alternatively we can express the scalar products $p_e \cdot p_e'$ etc. through so called Mandelstam variables, in particular for $2 \to 2$ processes. From

$$p_e + p_\mu = p_\mu' + p_e' \quad \Rightarrow \quad (p_e - p_e')^2 = (p_\mu - p_\mu')^2$$

we get the Mandelstam variables, defined as

$$\begin{aligned} t &\equiv (p_e - p_e')^2 = (p_\mu - p_\mu')^2, \\ s &\equiv (p_e + p_\mu)^2 = (p_e' + p_\mu')^2, \\ u &\equiv (p_e - p_\mu')^2 = (p_\mu - p_e')^2. \end{aligned}$$

We can say that we have

$$s + u + t = \sum_{i=1}^{4} m_i^2.$$

If we can neglect m_e^2, m_μ^2 against $p_e \cdot p_\mu$, we find that

$$\begin{aligned} t &= -2p_e \cdot p_e' = -2p_\mu \cdot p_\mu', \\ s &= 2p_e \cdot p_\mu = -2p_\mu' \cdot p_e', \\ u &= -2p_e' \cdot p_\mu = -2p_\mu' \cdot p_e. \end{aligned}$$

In this case we have that $s + u + t = 0$. We also find that in this limit we have

$$\frac{1}{\sqrt{(p_e \cdot p_\mu)^2 - m_e^2 m_\mu^2}} = \frac{2}{s}, \qquad \frac{1}{[(p_e - p_e')^2 + i\varepsilon]^2} = \frac{1}{t^2}.$$

If we express our result for σ in Mandelstam variables in the limit $m_e, m_\mu \to 0$ we have

$$\sigma = \int \frac{d^3 p_e'}{(2\pi)^2} \frac{\delta(E_e + E_\mu - E_e' - \sqrt{(\vec{p}_e + \vec{p}_\mu - \vec{p}_e')^2})}{E_e' E_\mu'} \frac{e^4}{2} \frac{2}{s} \frac{1}{t^2} \frac{s^2 + u^2}{4}.$$

By using the center of mass system with $\vec{p}_e + \vec{p}_\mu = 0$ our problem is only dependend on the angle θ. This is the so called scattering angle,

$$\begin{aligned} \sigma &= \int \frac{dp_e'(p_e')^2 d\cos\theta d\varphi}{4(2\pi)^2} \frac{\delta(E_e + E_\mu - p_e' - p_\mu')}{E_e' E_\mu'} e^4 \frac{s^2 + u^2}{st^2} = \\ &= \int_{-1}^{1} \frac{d\cos\theta}{16\pi} e^4 \frac{s^2 + u^2}{st^2}. \end{aligned}$$

By replacing $t = -2p_e' \cdot p_e = -2|\vec{p}_e'||\vec{p}_e|(1 - \cos\theta)$ we get that

$$dt = 2E_e^2 d\cos\theta, \qquad s = (p_e + p_\mu)^2 = 4E_e^2 \rightsquigarrow d\cos\theta = \frac{2}{s} dt.$$

So overall we see that we get

$$\frac{d\sigma}{dt} = 2\pi\alpha^2 \frac{s^2 + u^2}{s^2 t^2}, \tag{6.1}$$

with the fine-structure constant $\alpha = e^2/4\pi$. This is the so called differential cross section.

6.2 Compton-scattering

We will now calculate the process described by the two possible vertex combinations shown in figure 6.2. The second possibility can also be seen as $e^- e^+$-creation and $e^- e^+$-neutralization. We can see that Feynman diagrams do not have a time direction. With the diagram we can directly write

$$\begin{aligned} \mathcal{M} = \overline{u}(p_e', s_e') \quad & \left[(-ie\not{\epsilon}') \frac{i(\not{p}_e + \not{p}_\gamma + m)}{(p_e + p_\gamma)^2 - m^2 + i\varepsilon} (-ie\not{\epsilon}) + \right. \\ & \left. + (-ie\not{\epsilon}) \frac{i(\not{p}_e - \not{p}_\gamma' + m)}{(p_e - p_\gamma')^2 - m^2 + i\varepsilon} (-ie\not{\epsilon}') \right] u(p_e, s_e). \end{aligned}$$

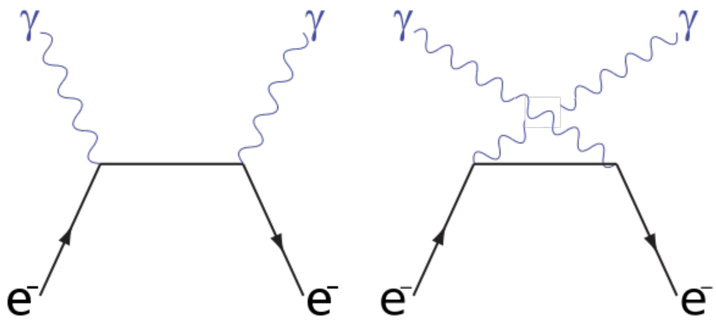

Figure 6.2: The two vertex combinations for Compton-scattering

For practical calculations it is necassary to choose a suited system. In this case we use the CMS with $\vec{p}_\gamma = -\vec{p}_e$. Due to gauge symmetry

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x),$$

we can pick $\epsilon^0 = 0$. Therefore we get

$$p_\gamma \cdot \epsilon = 0 = -\vec{p}_\gamma \cdot \vec{\epsilon} = \vec{p}_e \cdot \vec{\epsilon} = -p_e \cdot \epsilon.$$

The first relation is obtained by using the transversality, the other ones follow from the choice of our system. Therefore in this system and with this gauge we have

$$\epsilon \cdot p_e = \epsilon' \cdot p_e = 0.$$

The second one can be seen by picking

$$\begin{aligned} \epsilon_\mu \rightarrow \tilde{\epsilon}_\mu &= \epsilon_\mu - \frac{p_e \cdot \epsilon}{p_e \cdot p'_\gamma}(p'_\gamma)_\mu = \epsilon_\mu, \\ \epsilon'_\mu \rightarrow \tilde{\epsilon}'_\mu &= \epsilon'_\mu - \frac{p_e \cdot \epsilon'}{p_e \cdot p'_\gamma}(p'_\gamma)_\mu. \end{aligned}$$

Therefore we get

$$\frac{1}{2}\sum_{s_e, s'_e} |\mathcal{M}|^2 = e^4 \left[\frac{p_e p'_\gamma}{p_e p_\gamma} + \frac{p_e p_\gamma}{p_e p'_\gamma} + 4(\epsilon \cdot \epsilon')^2 - 2 \right].$$

With this equation we get the differential cross section of the Compton-scattering,

$$d^6\sigma = \frac{e^4}{4p_e \cdot p_\gamma}\left[\frac{p_e p'_\gamma}{p_e p_\gamma} + \frac{p_e p_\gamma}{p_e p'_\gamma} + 4(\epsilon \cdot \epsilon')^2 - 2\right]\delta^4(p_e + p_\gamma - p'_e - p'_\gamma)\frac{d^3p'_\gamma d^3p'_e}{16\pi^2 E'_\gamma E'_e}. \qquad (6.2)$$

The experimental process is shown in figure 6.3. In the experimental setup we search for $d\sigma/d\cos\theta$.

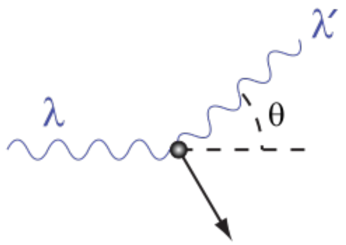

Figure 6.3: Experimental situation for the Compton-scattering

In the laboratory system we have

$$\vec{p}_e = \vec{0}, \quad p_e \cdot p_\gamma = p_e^0 p_\gamma^0 = m_e E_\gamma.$$

Therefore we find that

$$\vec{p}_\gamma \cdot \vec{p}'_\gamma = E_\gamma E'_\gamma \cos\theta, \qquad (\vec{p}_e + \vec{p}_\gamma \vec{p}'_\gamma)^2 = E_\gamma^2 + (E'_\gamma)^2 - 2E_\gamma E'_\gamma \cos\theta.$$

Since we do not detect the scattered electron we integrate over the $\int d^3\vec{p}'_e$ part. So we get that

$$\delta^3(\vec{p}_e + \vec{p}_\gamma - \vec{p}'_e - \vec{p}'_\gamma) \quad \Rightarrow \quad \vec{p}'_e = \vec{p}_e + \vec{p}_\gamma - \vec{p}'_\gamma.$$

This gives us $E'_e = \sqrt{(\vec{p}_e + \vec{p}_\gamma - \vec{p}'_\gamma)^2 + m^2}$. The remaining δ-distribution changed to

$$\delta(E_e + E_\gamma - E'_\gamma - \sqrt{E_\gamma^2 + (E'_\gamma)^2 - 2E_\gamma E'_\gamma \cos\theta + m^2}), \quad \Rightarrow \quad E'_\gamma = \frac{E_e E_\gamma}{E_\gamma(1 - \cos\theta) - E_e}.$$

We integrate over dE'_γ in $d^3p\gamma' = dE'_\gamma (E'_\gamma)^2 d\cos\theta 2\pi$ by using the root of the δ-argument and the derivative of the δ-distribution argument,

$$\frac{\partial(E_e + E_\gamma - E'_\gamma - \sqrt{E_\gamma^2 + (E'_\gamma)^2 - 2E_\gamma E'_\gamma \cos\theta + m^2})}{\partial E'_\gamma} = \frac{-2E_e - 2E_\gamma + 2E_\gamma \cos\theta}{2(E_e + E_\gamma - E'_\gamma)}.$$

In the end we get the Klein-Nishina equation (1929)

$$d\sigma = 2\pi d\cos\theta \frac{\alpha^2}{4}\frac{(E'_\gamma)^2}{E_\gamma^2 m_e^2}\left(\frac{E'_\gamma}{E_\gamma}+\frac{E_\gamma}{E'_\gamma}+4(\epsilon\cdot\epsilon')^2-2\right). \tag{6.3}$$

For not polarized photons we obtain that

$$\frac{1}{2}\sum_{\epsilon,\epsilon'}(\epsilon\cdot\epsilon')^2 = \frac{1}{2}(\cos^2\theta+1).$$

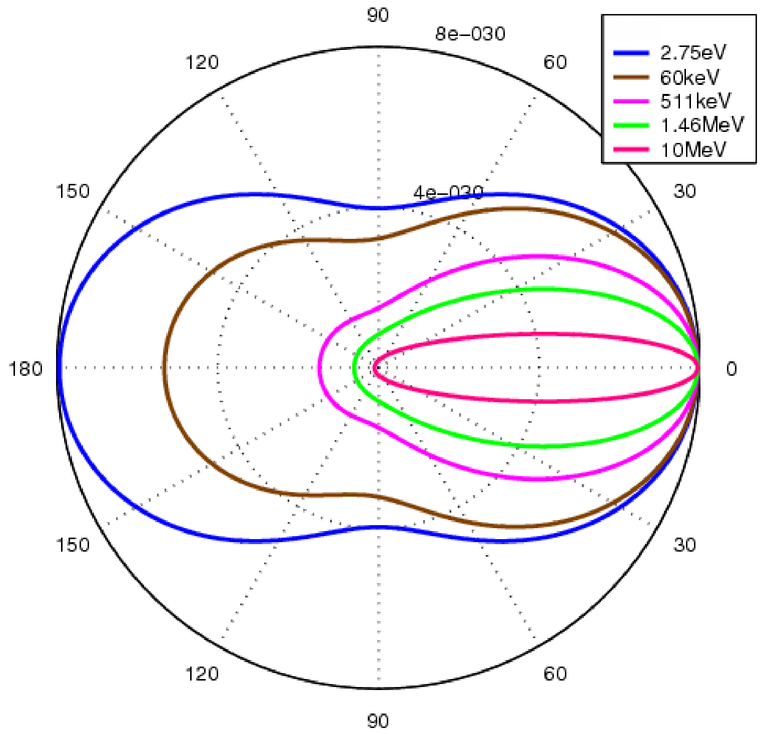

Figure 6.4: Klein-Nishina distribution of scattering-angle cross sections over a range of commonly encountered energies

7 Divergences, Pauli-Villars regularization, Renormalization

7.1 Introduction

We will now analyze the Feynman diagram shown in figure 7.1.

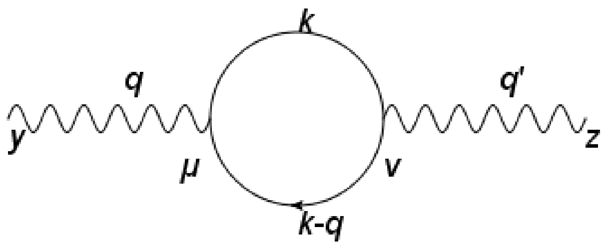

Figure 7.1: The Feynman diagram of the vacuum polarization process

Wick's theorem gives us

$$
\begin{aligned}
&\int d^4y d^4z \exp(iqy)\exp(-iq'z)\langle 0|T\left\{\hat{A}_\mu(y)\hat{A}_\nu(z)\frac{1}{2}e^2\right.\\
&\left.\cdot\int d^4x_1(-i)\hat{A}^\varrho(x_1)\hat{\bar{\psi}}_j(x_1)(\gamma_\varrho)_{jk}\hat{\psi}_k(x_1)\cdot\int d^4x_2(-i)\hat{A}^\lambda(x_2)\hat{\bar{\psi}}_l(x_1)(\gamma_\lambda)_{lm}\hat{\psi}_m(x_2)\right\}|0\rangle =\\
= \;&(2\pi)^4\delta^4(q-q')\frac{-i}{q^2+i\varepsilon}\frac{-i}{q'^2+i\varepsilon}\underbrace{\left[-e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\left\{\frac{1}{\not{k}-\not{q}-m+i\varepsilon}\gamma_\mu\frac{1}{\not{k}-m+i\varepsilon}\gamma_\nu\right\}\right]}_{\equiv I_{\mu\nu}}.
\end{aligned}
$$

We call $I_{\mu\nu}$ the vaccum polarization. We used that

$$
\frac{\not{k}+m}{k^2-m^2+i\varepsilon}=\frac{1}{\not{k}-m+i\varepsilon},\qquad \frac{\not{k}-\not{q}+m}{(k-q)^2-m^2+i\varepsilon}=\frac{1}{\not{k}-\not{q}-m+i\varepsilon}.
$$

We see directly that $I_{\mu\nu}$ is quadratic divergent. Thus every amplitude is divergent, e.g. the electron-myon-scattering process we analyzed in the last chapter. In general it is possible to integrate a divergent loop (vaccum polarization) in every (even simple) Feynman diagram (just in a photon line). Therefore we need a rule how to handle something like that well defined - or the theory fails here.

The physical reason for this divergence is that graviation is neglected in QFT. Therefore the QFT is not applicable for all problems. If we compare the Coloumb-Law with the Law of Graviation, we see that

$$V_C = \alpha \frac{Q_1 Q_2}{r}, \qquad V_G = G \frac{M_1 M_2}{r} = G \frac{E_1 E_2}{r}.$$

For energies $E^2 > 1/G$ the gravitation is so strong, that it cannot be treated pertubativly any more. Since the elementary particle twists the space, another particle will see a twisted space on the Planck scale. The necessary energy is then

$$E_{pl} = \frac{1}{\sqrt{G}} = 1.22 \cdot 10^{19} \text{ GeV}.$$

We can also set the Planck length based on the Planck energy, $l_{pl} = 1/E_{pl}$. Therefore we demand that we accept only theories which uncouple the physics from the Planck scale.

We will see that this happens if there are maximal logarithmical divergences. Let Λ be for instance M_{pl} (Planck mass/energy). With two experiments we find

$$\left.\begin{array}{ll} \text{Experiment 1}: & \propto \ln \frac{q_1^2}{\Lambda^2}, \\ \text{Experiment 2}: & \propto \ln \frac{q_2^2}{\Lambda^2}. \end{array}\right\} \Delta \propto \ln \frac{q_1^2}{q_2^2}.$$

We see that this is finite and independent of Λ. To achieve this we have a execute two steps.

1. **Regularization**

   This is the parametrization of the divergence $\rightarrow \ln q^2 - \ln \Lambda^2$. There are several ways to do this.

   - *Pauli-Villars regularization* Here we introduce a new (not physical) particle with $M \approx E_{pl}$. We get

$$\ln \frac{q^2}{M^2} \quad \Rightarrow \quad \Lambda = M.$$

One advantage of this technique is that is explicit Lorentz invariant.

- *Cutoff regularization* In this technique the integral will be cut,

$$\int_0^\Lambda d\sqrt{k^2} \cdots \to \ln \frac{q^2}{\lambda^2}.$$

The big disadvantage of this technique is that it is not Lorentz invariant in general.

- *Lattice regularization* We replace the spacetime with a discrete lattice, whose points have the constant distance of a. We see that $a = 1/\Lambda$. Therefore we have a technique which is mathematically very clean and a could be $\approx l_{pl}$.

- *Dimensional regularization* This is the standard technique (called $\overline{\mathrm{MS}}$). In this technique we have four elementary steps:

  a) Via Wick rotations we transform the metric into an euclidian metric. We make the analytic continuation of $g_{\mu\nu}$ to

$$g^E_{\mu\nu} = -\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \int_{-\infty}^{\infty} dk^0 \to -\int_{i\infty}^{-i\infty} dk^0 = \int_{-\infty}^{\infty} dk^0_E.$$

  b) We generalize for $d \in \mathbb{N}$, with d being the spacetime dimension ($d = 4$).

  c) We make the analytic continuation for $d \in \mathbb{C}$ with some constraints.

  d) We identify the poles in ε for $d = 4 - 2\varepsilon$,

$$\frac{1}{\varepsilon}\left(\frac{q^2}{\mu^2}\right)^\varepsilon = \frac{1}{\varepsilon}\exp\left(\varepsilon \ln\left[\frac{q^2}{\mu^2}\right]\right) = \frac{1}{\varepsilon} + \ln\frac{q^2}{\mu^2}.$$

  This technique is also Lorentz invariant.

2. **Renormalization**

Here we put everything together in our elementary charge. We get

$$e^2 \to e^2_R(q_0^2) = e^2\left[1 + \cdots \ln\left(\frac{q^2}{\mu^2}\right)\right].$$

7.2 The contributions

The divergent contributions

- Vacuum polarization:

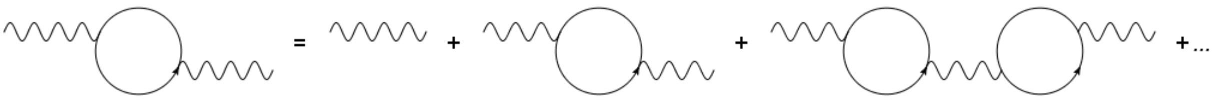

Figure 7.2: The series of Feynman diagrams of the vacuum polarization process

This is like

$$\begin{aligned} & \left[g_{\mu\lambda} + \left(-\frac{i\alpha}{3\pi}\right)\ln\frac{M^2}{m^2}(q^2 g_{\mu\nu} - q_\mu q_\nu)\frac{-ig^\nu{}_\lambda}{q^2+i\varepsilon} + \mathcal{O}(\alpha^2)\right] = \\ = \;& \underbrace{\left[1 - \frac{\alpha}{3\pi}\ln\frac{M^2}{m^2}\right]}_{\equiv Z_3} + \mathcal{O}(\alpha^2). \end{aligned}$$

- Self energy:

Figure 7.3: The series of Feynman diagrams of the self energy process

This is like

$$\begin{aligned} & \frac{i}{\not p - m + i\varepsilon}\left[1 + i\left(\frac{\alpha}{4\pi}3m\ln\frac{M^2}{m^2} - (\not p - m)\frac{\alpha}{4\pi}\ln\frac{M^2}{m^2}\right) i\frac{\not p + m}{p^2 - m^2 + i\varepsilon} + \mathcal{O}(\alpha^2)\right] = \\ = \;& \frac{i}{\not p - m + i\varepsilon}\left[1 - \frac{\alpha}{4\pi}\ln\frac{M^2}{m^2}\right]\left[1 + \frac{3\alpha}{4\pi}m\ln\frac{M^2}{m^2}\frac{\not p + m}{p^2 - m^2 + i\varepsilon}\right] + \mathcal{O}(\alpha^2) = \\ = \;& \left[1 - \frac{\alpha}{4\pi}\ln\frac{M^2}{m^2}\right]\frac{i}{\not p - m + i\varepsilon}\frac{1}{1 - \frac{3\alpha}{4\pi}m\ln\frac{M^2}{m^2}\frac{\not p + m}{p^2 - m^2 + i\varepsilon}} + \mathcal{O}(\alpha^2) = \\ = \;& \underbrace{\left[1 - \frac{\alpha}{4\pi}\ln\frac{M^2}{m^2}\right]}_{\equiv Z_2^{-1}}\frac{i}{\not p - m\left(1 + \frac{3\alpha}{4\pi}\ln\frac{M^2}{m^2}\right) + i\varepsilon}. \end{aligned}$$

We say that $m_R = m\left(1 + \frac{3\alpha}{4\pi}\ln\frac{M^2}{m^2}\right)$ is the renormed mass. We absorb this into

$$\overline{\psi}(x)\left[Z_2(\not{p} - m) - e_R\not{A}\right]\psi(x).$$

- Vertex correction:

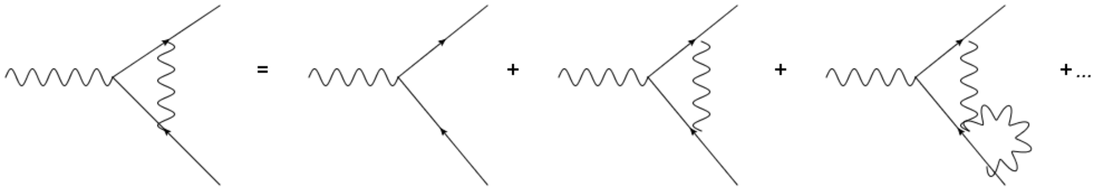

Figure 7.4: The series of Feynman diagrams of the vertex correction process

We get

$$e_R^{VC} = \left(1 - \frac{\alpha}{4\pi}\ln\frac{M^2}{m^2}\right)e = Z_1 e.$$

We can now calculate that

$$Z_1 Z_2 = \left(1 - \frac{\alpha}{4\pi}\ln\frac{M^2}{m^2}\right)\left(1 + \frac{\alpha}{4\pi}\ln\frac{M^2}{m^2}\right) = 1 + \mathcal{O}(\alpha^2).$$

We will proof in quantum field theory that

$$S_F^{-1}(p) = -i(\not{p} - m) + i\varepsilon, \qquad S_F^{-1}(p+q) = -i(\not{p} + \not{q} - m) + i\varepsilon.$$

Therefore we obtain that

$$S_F^{-1}(p+q) - S_F^{-1}(p) = -i\not{q} = (-ie\gamma_\mu)q^\mu\frac{1}{e} = \Gamma_\mu\frac{q^\mu}{e},$$

with the general vertex function Γ_μ. It is possible to show that this is valid in all orders since it is obtained from the gauge invariance. The proof requires non pertubative methods - like generating functions. We see that the renormalized vertex is inverse to the renormalization of the propagator. This is the so called *Ward-Identity*.

Since S_F^{-1} is only a two point function (propagator) and Γ_μ is a three point function we see directly that we have the renormalization for an arbitrary point function. Therefore everything is renormalized!

The renormalization itself is in the way that we split everything. We see that in the end we have

$$e_R = e_0 \left(1 - \frac{\alpha}{6\pi} \ln \frac{M^2}{m^2}\right) + \mathcal{O}(\alpha^2) = e_0 \sqrt{Z_3}.$$

The finite momentum-dependent contributions give us measurable effects.

7.3 Generalization on arbitrary theories

Our kind of proof requires that we have at most logarithmic divergences. Which theories could fulfill this requirement? We take a look at

$$S = \int d^4x \overline{\psi}(x)(\hat{\not{p}} - m)\psi(x),$$

where we have the units (with $\hbar = 1$) d^4x length^4 and energy^4 for the rest (means no unit at all). Therefore we see that $\psi(x)$ has the dimension energy $^{3/2}$. For bosons we see that

$$S_{\text{boson}} = \int d^4x \Phi^*(x)(\hat{p}^2 - m^2)\Phi(x),$$

so that $\Phi(x)$ has the dimension energy. We can now see that there cannot be a coupling constant with negative mass dimension, like

$$\mathcal{L}_{\text{int}} = g\overline{\psi}(x)A_\mu(x)A^\mu(x)\psi(x).$$

Therefore the graph in figure 7.5 would be divergent like Λ^7. So only coupling on one field is possible.

We can take another look at this by just supposing that there is a vertex like the first one in figure 7.6. Then the 2nd one of this figure would also be possible and the 3rd one as well. But with every new graph we would have to introduce there would be another one possible.

Therefore we see that we lose all predictive power for our theory, because we would require infinity couplings with infinity coupling constants. So g is only allowed to couple on a single field. *We say that a theory is called renormalizable, if there is only a finite number of counter terms.*

It is only allowed to have three combinations for interactions:

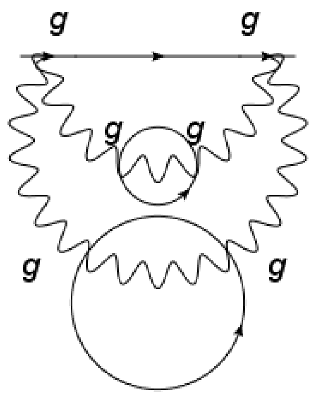

Figure 7.5: A graph which could only exist when there would be a coupling on more than one field

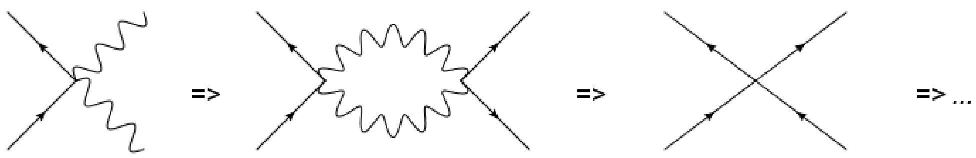

Figure 7.6: Other possible Feynman diagrams if there would be more than one field coupling possible

- 2 Fermions and 1 Boson Vertex,
- Three Bosons Vertex or
- Four Bosons Vertex like

$$\mathcal{L}_{\text{Higgs}} = \frac{\lambda}{4i} \left(\Phi^*(x)\Phi(x)\right)^2 .$$

The possible divergences are:

- Fermion loops like the ones shown in figure 7.7.

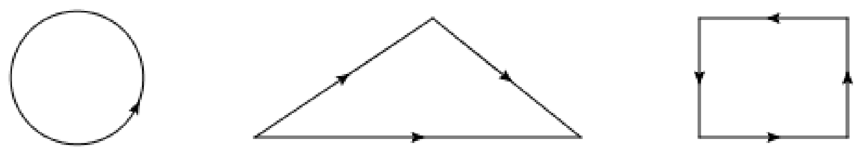

Figure 7.7: The Fermion loop divergences - quadratically, linear and logarithmic

While the first one is pot. quadratically divergent, we obtain from gauge invariance that it must be $\log(\Lambda, M)$. The second one seems to be a problem and will be discussed next (pot. linear divergent). The third one is always $\log(\Lambda)$.

- Boson loop from figure 7.8 is always $\log(\Lambda)$ and therefore simple.

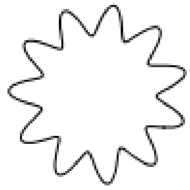

Figure 7.8: The Boson loop divergence

- Mixed loops from figure 7.9 is always $\log(\Lambda)$ as well.

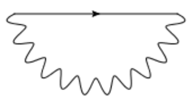

Figure 7.9: The mixed loop divergence

The proof for the second fermion loop (figure 7.7, graph 2) is based on the fact that in quantum electrodynamics this graph and its reversed one is always zero. The reversed one is calculated by charge conjugation ($\hat{C} = C\text{c.c.}$). We get

$$I = \mathrm{tr}\left\{(-ie\gamma_{\mu_1})C^{-1}CS_F(x_1 - x_2)C^{-1}C(-ie\gamma_{\mu_2})C^{-1}CS_F(x_2 - x_3)...\right\},$$

where we see that

$$\begin{aligned} CS_F(x_1 - x_2)C^{-1} &= S_F^T(x_2 - x_1), \\ C(-ie\gamma_{\mu_2})C^{-1} &= (-i\gamma^2\gamma^0)\gamma_{\mu_2}(-i\gamma^2\gamma^0) = -\gamma_{\mu_2}^T. \end{aligned}$$

So the trace is after all zero. While it is often possible to just use the complex conjugated term for the reversed graph we must be quite careful to use it all the time. E.g. in the weak interaction we have a graph with $\gamma_{\mu_1}, \gamma_{\mu_2}$ and $\gamma_{\mu_3}\gamma_5$. Here just taking the complex conjugate won't us help at all to reverse the graph!

7.4 Infrared divergences

We are looking at a charge Q with an electric field E. By lorentz transformation (boost) we obtain that all (virtual) photons are now real photons with $q^2 = 0$. This can be described by a distribution of photons, with a probability $\propto \omega^{-1}$ for small ω. We use this to investigate the **Bremsstrahlung**.

At every change of momentum we find Bremsstrahlung for $|\vec{v}| \approx c$. We have the calculate the Feynman diagrams shown in figure 7.10.

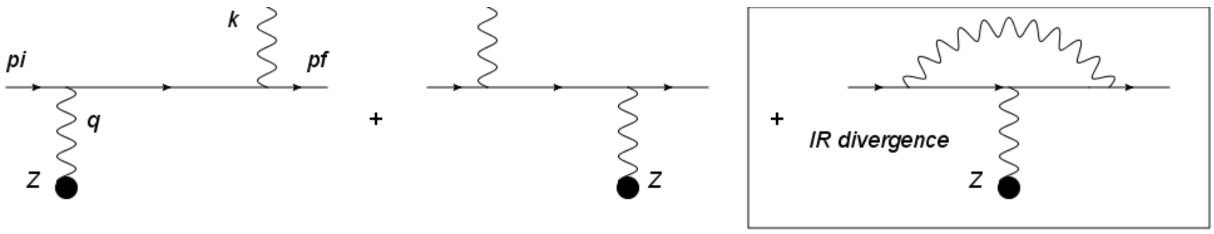

Figure 7.10: The important Feynman diagrams for the Bremsstrahlung

Remark We quantized in a way that $\langle e\gamma|e\rangle \equiv 0$. This is not very physical, but has some advantages. In this case we have to deal with some divergences, but this disadvantage can be used to cross-check the result. If all divergences vanish we have made now mistake in our calculations (mostly).

We replace (for the atom in the rest frame)

$$A^\mu(x) \to A^\mu_{\text{plain wave}}(x) + A^0_{\text{coulomb}}(x).$$

The cross-section is then given by

$$\sigma = \int \frac{d^3k d^3p_f}{(2\pi)^6} 2\pi\delta(E_f + k_0 - E_i) \frac{Z^2 e^6}{2k^0 E_f E_i |\vec{v}_i|} \frac{1}{(|\vec{p}_f - \vec{p}_i + \vec{k}|^2)^2} |\mathcal{M}|^2,$$

with the already inserted $(2\pi)^3\delta^3(\vec{q} - \vec{p}_f + \vec{p}_i - \vec{k})$ and the photon propagator

$$\frac{1}{q^2 + i\varepsilon} = \frac{1}{(|\vec{p}_f - \vec{p}_i + \vec{k}|^2)^2}.$$

We remember this Feynman diagram from the Compton-scattering. We have

$$
\begin{aligned}
|\mathcal{M}|^2 \quad &= \quad \frac{1}{2}\sum_{\epsilon}\sum_{s_i,s_f}\left|\overline{u}(p_f,s_f)\left[(-i\not\epsilon\frac{i}{\not p_f+\not k-m+i\varepsilon}(-i\gamma_0)+\right.\right. \\
&\left.\left. +(-i\gamma_0)\frac{i}{\not p_i-\not k-m+i\varepsilon}(-i\not\epsilon)\right]u(p_i,s_i)\right|^2 .
\end{aligned}
$$

After some elementary steps we obtain (compare Compton-scattering)

$$
\sigma = \frac{1}{2}\sum_{s_i,s_f}\int \frac{Z^2e^6m^2}{2k^0E_fE_i|\vec{v}|}\frac{1}{(q^2)^2}2\pi\delta(E_f+k^0-E_i)\left|\overline{u}(p_f)\gamma_0 u(p_i)\right|^2\sum_{\epsilon}\left|\frac{p_f\cdot\epsilon}{p_f\cdot k}-\frac{p_i\cdot\epsilon}{p_i\cdot k}\right|^2\frac{d^3kd^3p_f}{(2\pi)^6}.
$$

We are interested in a special case - the limit $k^0 \to 0$. In this limit we have $|\vec{p}_f| \to |\vec{p}_i|$ with $E_f \to E_i$ since we have only a change in the direction - not in the momentum, since we have a very low momentum transfer. Finally the task is reduced on the calculation of the integral

$$
I = -\int\frac{d^3k}{(2\pi)^3}\frac{e^2}{2k^0}\left(\frac{m^2}{(p_f\cdot k)^2}+\frac{m^2}{(p_i\cdot k)^2}-2\frac{p_ip_f}{p_f\cdot kp_i\cdot k}\right).
$$

In the limit $\vec{p}_f = \vec{p}_i$ we have $I = 0$. Therefore we make a Taylor expansion in $q_\mu = p_{f,\mu} - p_{i,\mu}$. This is supported by the fact that infrared divergences only appear for $\vec{p}_f \to \vec{p}_i$. Our series will look like

$$
I = B^\mu q_\mu + \frac{1}{2}C^{\mu\nu}q_\mu q_\nu + ...
$$

First of all we insert $p_f = q + p_i$ in the original term and obtain that

$$
I = -\int\frac{d^3k}{(2\pi)^3}\frac{e^2}{2k^0}\left(\frac{m^2}{(p_i\cdot k+q\cdot k)^2}+\frac{m^2}{(p_i\cdot k)^2}-2\frac{m^2+q\cdot p_i}{(p_i\cdot k+q\dot{k})(p_i\cdot k)}\right).
$$

For the first factor in the taylor series we obtain that

$$
B^\mu = -\int_0^{k_{\min}}\frac{d^3k}{2k^0}e^2\left(-2\frac{m^2k^\mu}{(p_i\cdot k+q\cdot k)^3}+0+2\frac{m^2k^\mu}{(p_i\cdot k)^3}-2\frac{p_i^\mu}{(p_i\cdot k)^2}\right).
$$

We see that this is logarithmic divergent, because the first three terms are zero for $q = 0$. For the upper limit of the integral we set $k_{\min}$. This is a threshold for the solution of our detector, i.e. for $|\vec{k}| < k_{\min}$ we cannot detect the photon. The important thing is that if we set k_{min} to 0 (or nearly at zero) we have to consider all (a lot) higher order terms to be correct.

For practical calculations we use

$$k^0 = \lambda \cosh(s), \qquad |\vec{k}| = \lambda \sinh(s).$$

We are now looking at the limit $\lambda \to 0$ and obtain that $s \to \infty$. Therefore our result looks like

$$B^\mu = 2\pi \frac{\alpha}{2\pi^2} p_i^\mu \int_0^{\operatorname{arcsinh}(k_{\text{min}}/\lambda)} ds \frac{\sinh^2(s)}{m^2} \int_{-1}^{1} d\cos\vartheta \frac{1}{(\cosh(s)\cosh(t) - \sinh(s)\sinh(t)\cos\vartheta)^2},$$

because $\operatorname{arcsinh}(k_{\text{min}}/\lambda)$ goes to $\ln z + \ln 2$ for $k_{\text{min}}/\lambda \to \infty$. Overall we have found that

$$B^\mu = \frac{2\alpha}{\pi} \frac{p_i^\mu}{m^2} \ln \frac{k_{\text{min}}}{\lambda} + ...$$

We see directly that for $\lambda \to 0$ we have a divergent term in form of $\ln x$ with $x \to \infty$. But this contribution has the same shape as the contribution from the vertex correction,

$$\frac{2\alpha}{3\pi} \frac{q^2}{m^2} \ln \frac{m}{\lambda}.$$

From $C^{\mu\nu}$ we get also a contribution which looks quite familiar and with the usage of

$$q_\mu \cdot p_i^\mu = \frac{1}{2} (p_f - p_i) \left((p_f + p_i) - (p_f - p_i) \right),$$

we obtain a finite result without divergences for $\lambda \to \infty$,

$$\frac{2\alpha}{3\pi} \frac{q^2}{m^2} \ln \frac{m}{k_{\text{min}}}.$$

8 Vertex function, Vacuum polarization and Self-energy

8.1 Vacuum polarization

We will now calculate

$$\Delta I_{\mu\nu} = -4e^2 \int \frac{d^4k}{(2\pi)^4} \left[\frac{(k-q)_\mu \cdot k_\nu + (k+q)_\nu \cdot k_\mu - g_{\mu\nu}(k\cdot(k-q)-^2)}{[(k-q)^2 - m^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]} \right].$$

To summarize this introduction a renormalizable theory is a theory where physical observables do not depend on M at laboratory energies.

Remark If the quadratic divergence does not vanish the gauge invariance will be broken. We calculate

$$\begin{aligned} q^\mu I_{\mu\nu} &= -e^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{tr} \left\{ \frac{1}{\not{k} - \not{q} - m + i\varepsilon} \left((\not{k} - m + i\varepsilon) - (\not{k} - \not{q} - m + i\varepsilon) \right) \frac{1}{\not{k} - m + i\varepsilon} \gamma_\nu \right\} = \\ &= -e^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{tr} \left\{ \frac{1}{\not{k} - \not{q} - m + i\varepsilon} \gamma_\nu - \frac{1}{\not{k} - m + i\varepsilon} \gamma_\nu \right\}. \end{aligned}$$

If we make a substitution $k_\mu \to k_\mu + q_\mu$ in the first term we get 0 as it should be. But we cannot do this, because it is not allowed to shift the integrand in divergent integrals. This is due to the surface terms. One example:

$$\begin{aligned} \lim_{\Lambda\to\infty} \quad & \int_{-\Lambda}^{\Lambda} dx \left((x-a)^2 - x^2 \right) = \lim_{\Lambda\to\infty} \left[\int_{-\Lambda-a}^{\Lambda-a} dx x^2 - \int_{-\Lambda}^{\Lambda} dx x^2 \right] = \\ &= \lim_{\Lambda\to\infty} \left[\frac{1}{3}(\Lambda - a)^3 + \frac{1}{3}(\Lambda + a)^3 - (\frac{1}{3}\Lambda^3 + \frac{1}{3}\Lambda^3) \right] = \lim_{\Lambda\to\infty} 2\Lambda a^2 = \infty. \end{aligned}$$

To continue the calculation on $I_{\mu\nu}$ we come to the trace of

$$-e^2 \int \frac{d^4k}{(2\pi)^4} \left[\frac{\mathrm{tr}\{(\not{k} - \not{q} + m)\gamma_\mu(\not{k} + m)\gamma_\nu\}}{[(k-q)^2 - m^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]} - (m \to M) \right].$$

By using our trace identities we obtain that

$$\mathrm{tr}\{\cdots\} = 4 \left(m^2 g_{\mu\nu} + (k-q)_\mu k_\nu - k(k+q) g_{\mu\nu} + (k-q)_\nu k_\mu \right).$$

We will now introduce a trick for calculating such integrals, called *Feynman-parameters.*

1. The main idea is

$$\frac{i}{k^2 - m^2 + i\varepsilon} = \int_0^\infty dz \exp\left(iz(k^2 - m^2 + i\varepsilon) \right).$$

2. We can use this to get

$$\frac{1}{A_1 A_2 \cdots A_n} = (n-1)! \int_0^1 dz_1 \cdots \int_0^1 dz_n \frac{\delta(1 - z_1 - z_2 - ... - z_n)}{(A_1 z_1 + A_2 z_2 + ... + A_n z_n)^n}.$$

We will use the first trick to simplify our solution. We get

$$\begin{aligned} I_{\mu\nu} = \quad & 4e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^\infty dz_1 \int_0^\infty dz_2 \left[\exp(iz_1(k^2 - m^2 + i\varepsilon) + iz_2((k-q)^2 - m^2 + i\varepsilon)) \cdot \right. \\ & \left. \cdot \left((k-q)_\mu k_\nu + (k-q)_\nu k_\mu - g_{\mu\nu}(k \cdot (k-q) -^2)\right) - (m \to M) \right]. \end{aligned}$$

We now substitute $k_\mu \to l_\mu + q_\mu \frac{z_2}{z_1 + z_2}$. This is well defined if the integrals commutate, which is always the case when the integrals exist. Therefore we obtain

$$\begin{aligned} iz_1(k^2 - m^2 + i\varepsilon) \quad & + \quad iz_2((k-q)^2 - m^2 + i\varepsilon) = iz_1(l^2 - m^2 + i\varepsilon) + 2iz_1 ql \frac{z_2}{z_1 + z_2} + \\ & + \quad iz_1 q^2 \frac{z_2^2}{(z_1+z_2)^2} + iz_2(l^2 - m^2 + i\varepsilon) - 2iz_2 ql \frac{z_1}{z_1+z_2} + iz_2 q^2 \frac{z_1^2}{(z_1+z_2)^2}. \end{aligned}$$

We see that all mixed terms vanish. Finally we have

$$i(z_1 + z_2)(l^2 - m^2 + i\varepsilon) + iq^2 \frac{z_1 z_2}{z_1 + z_2}.$$

This means for our integral that

$$
\begin{aligned}
I_{\mu\nu} = \quad & 4e^2 \int_0^\infty dz_1 \int_0^\infty dz_2 \int_{-\infty}^\infty \frac{d^4 l}{(2\pi)^4} \Big\{ \exp(i(z_1+z_2)(l^2 - m^2 + i\varepsilon) + iq^2 \frac{z_1 z_2}{z_1+z_2}) \cdot \\
& \cdot \left(\left(l_\mu - q_\mu \frac{z_1}{z_1+z_2} \right) \left(l_\nu + q_\nu \frac{z_2}{z_1+z_2} \right) + \left(l_\nu - q_\nu \frac{z_1}{z_1+z_2} \right) \left(l_\mu + q_\mu \frac{z_2}{z_1+z_2} \right) - \right. \\
& \left. - \, g_{\mu\nu} \left(\left(l + q \frac{z_2}{z_1+z_2} \right) \left(l - \frac{z_1}{z_1+z_2} \right) - m^2 \right) \right) - (m \to M) \Big\}
\end{aligned}
$$

We see that $\int d^4 l \cdots l_\mu l_\nu$ is only $\neq 0$ for $\mu = \nu$. Therefore we set

$$
\propto \int d^4 l g_{\mu\nu} l^2 \quad \rightsquigarrow \quad \propto d^4 l 4 l^2, \qquad l_\mu l_\nu \to \frac{1}{4} g_{\mu\nu} l^2 .
$$

In the end we obtain

$$
\begin{aligned}
I_{\mu\nu} = \quad & 4e^2 \int_0^\infty dz_1 \int_0^\infty dz_2 \int \frac{d^4 l}{(2\pi)^4} \Big\{ \exp\left(i(z_1+z_2)(l^2 - m^2 + i\varepsilon) + iq^2 \frac{z_1 z_2}{z_1+z_2} \right) \cdot \\
& \cdot \left[g_{\mu\nu} \frac{2}{4} l^2 - 2 q_\mu q_\nu \frac{z_1 z_2}{(z_1+z_2)^2} - g_{\mu\nu} l^2 + g_{\mu\nu} q^2 \frac{z_1 z_2}{(z_1+z_2)^2} + m^2 g_{\mu\nu} \right] - (m \to M) \Big\} .
\end{aligned}
$$

To calculate the integrals we need to know the result of a Gaussian integral,

$$
\int_{-\infty}^\infty \frac{dl}{2\pi} \exp(i l^2 (a + i\eta)) = \frac{\exp(-i\pi/4)}{2\sqrt{\pi a}} .
$$

With the help of this identity we get

$$
\int_{-\infty}^\infty \frac{dl}{2\pi} l^2 \exp(i l^2 (a + i\eta)) = \frac{i \exp(-i\pi/4)}{4a\sqrt{\pi a}} .
$$

We now have for our $I_{\mu\nu}$ that

$$
\begin{aligned}
I_{\mu\nu} = \quad & 4e^2 \int_0^\infty dz_1 \int_0^\infty dz_2 \left(\exp\left(iq^2 \frac{z_1 z_2}{z_1+z_2} - i(z_1+z_2)(m^2 - i\varepsilon) \right) \frac{\exp(-i\pi/2)}{16\pi^2 (z_1+z_2)^2} \cdot \right. \\
& \left. \cdot \left[- \frac{4i}{2(z_1+z_2)} \frac{1}{2} g_{\mu\nu} - 2 q_\mu q_\nu \frac{z_1 z_2}{(z_1+z_2)^2} + g_{\mu\nu} q^2 \frac{z_1 z_2}{(z_1+z_2)^2} + g_{\mu\nu} m^2 \right] - (m \to M) \right) .
\end{aligned}
$$

By doing a short calculation we can see that

$$A = \int_0^\infty dz_1 \int_0^\infty dz_2 \frac{1}{(z_1+z_2)^2}\left(m^2 - \frac{i}{z_1+z_2} - \frac{q^2 z_1 z_2}{(z_1+z_2)^2}\right)\exp(i(\cdots)) = 0.$$

To proof this we just have to scale $z_i \to \lambda z_i$. This results in

$$\begin{aligned} A &= \int_0^\infty dz_1 \int_0^\infty dz_2 \frac{1}{(z_1+z_2)^2}\left[m^2 - \frac{i}{\lambda(z_1+z_2)} - q^2\frac{z_1 z_2}{(z_1+z_2)^2}\right]\exp(i\lambda(\cdots)) = \\ &= i\lambda\frac{\partial}{\partial\lambda}\left[\int_0^\infty dz_1 \int_0^\infty dz_2 \frac{1}{\lambda(z_1+z_2)^3}\exp(i\lambda(\cdots))\right]. \end{aligned}$$

We see that by undoing the scaling ($\lambda z_i \to z_i$) we obtain that

$$A = i\lambda\frac{\partial}{\partial\lambda}\left(\int_0^\infty dz_1 \int_0^\infty dz_2 \frac{1}{(z_1+z_2)^3}\exp(i(\cdots))\right) = 0.$$

Using this we obtain that

$$\begin{aligned} I_{\mu\nu} = \quad & -2i\frac{\alpha}{\pi}(q^2 g_{\mu\nu} - q_\mu q_\nu)\int_0^\infty dz_1 \int_0^\infty dz_2 \left[\exp(iq^2\frac{z_1 z_2}{z_1+z_2} - \right. \\ & \left. -i(z_1+z_2)(m^2 - i\varepsilon))\frac{z_1 z_2}{(z_1+z_2)^4} - (m \to M)\right]. \end{aligned}$$

The projection term $(q^2 g_{\mu\nu} - q_\mu q_\nu)$ appears quite often in gauge fields. Therefore we get $q^\mu I_{\mu\nu} = 0$ and $q^\nu I_{\mu\nu} = 0$. We now can use another trick. We will insert

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{z_1+z_2}{\lambda}\right), \qquad z_1 \to \lambda w_1, z_2 \to \lambda w_2.$$

Therefore we obtain an integral which can easily be solved using

$$\int_0^\infty \frac{d\lambda}{\lambda}(\exp(i\lambda(a+i\varepsilon)) - \exp(i\lambda(b+i\varepsilon))) = \ln\left[\frac{b}{a}\right].$$

So the temporary result can be written with the help of $\ln b/a = \ln b/m^2 + \ln m^2/a$ as

$$\begin{aligned} I_{\mu\nu} &= -2i\frac{\alpha}{6\pi}(q^2 g_{\mu\nu} - q_\mu q_\nu)\ln\frac{M^2}{m^2} + \\ &+ 2i\frac{\alpha}{\pi}(q^2 g_{\mu\nu} - q_\mu q_\nu)\int_0^1 dz_1 z_1(1-z_1)\ln\left[1 - \frac{q^2}{m^2}z_1(1-z_1)\right]. \end{aligned}$$

We now have to work on the integral

$$
\begin{aligned}
I &= \frac{2\alpha}{\pi}\int_0^1 dz z(1-z)\ln\left[1-\frac{q^2}{m^2}z(1-z)\right] = \\
&= \frac{2\alpha}{\pi}\left(\frac{z^2}{2}-\frac{z^3}{3}\right)\ln\left[1-\frac{q^2}{m^2}z(1-z)\right]\Bigg|_0^1 - \frac{2\alpha}{\pi}\int_0^1 dz\left(\frac{z^2}{2}-\frac{z^3}{3}\right)\frac{-(q^2/m^2)(1-2z)}{1-(q^2/m^2)z(1-z)}.
\end{aligned}
$$

Through substitution $2z-1=w$ we obtain

$$
\begin{aligned}
I &= -\frac{\alpha}{\pi}\int_{-1}^1 dw\frac{(w+1)^2}{4}\left(\frac{1}{3}-\frac{w}{6}\right)\frac{wq^2/m^2}{1-(1-w^2)q^2/4m^2} = \\
&= -\frac{\alpha}{6\pi}\int_{-1}^1 dw\frac{w^2(3-w^2)}{(w-\sqrt{1-4m^2/q^2+i\eta})(w+\sqrt{1-4m^2/q^2+i\eta})}.
\end{aligned}
$$

This is a scenario where the residue theorem fits. We require a case-by-case analysis. We set $\zeta \equiv 1-4m^2/q^2+i\eta$ and rewrite

$$
\begin{aligned}
I &= -\frac{\alpha}{6\pi}\int_{-1}^1 dw\frac{((w^2-\zeta)+\zeta)(3-\zeta+(\zeta-w^2))}{(w-\sqrt{\zeta})(w+\sqrt{\zeta})} = \\
&= \frac{-\alpha}{6\pi}\int_{-1}^1 dw(3-w^2)+\frac{\alpha}{6\pi}\zeta\int_{-1}^1 dw - \frac{\alpha}{6\pi}\zeta(3-\zeta)\int_{-1}^1\frac{dw}{(w-\sqrt{\zeta})(w+\sqrt{\zeta})}.
\end{aligned}
$$

To solve the last (non-trivial) integral we investigate:

- The range from $-\infty < q^2 < 0$. The solution is

$$
\frac{1}{2}\frac{1}{\sqrt{\zeta}}\ln\left|\frac{w-\sqrt{\zeta}}{w+\sqrt{\zeta}}\right|, \qquad \zeta > 0.
$$

- The range from $0 \le q^2 \le 4m^2$. The solution here is

$$
\frac{1}{\sqrt{-\zeta}}\arctan\frac{w}{\sqrt{-\zeta}}, \qquad \zeta < 0.
$$

- In the range from $4m^2 < q^2$ we get terms like

$$
\frac{1}{w-\sqrt{\xi}-i\eta} = \underbrace{\mathrm{PV}\left(\frac{1}{w-\sqrt{\xi}}\right)}_{\equiv B} + i\pi\delta(w-\sqrt{\xi}),
$$

  with $\xi \equiv 1-4m^2/q^2$.

Calculating the principle value we get

$$
\begin{aligned}
B &= \lim_{\alpha\to 0}\left[\int_0^{\sqrt{\xi}-\alpha} + \int_{\sqrt{\xi}+\alpha}^{1}\right] dw \frac{1}{w^2-\xi} = \\
&= \lim_{\alpha\to 0}\left[\frac{1}{2}\frac{1}{\sqrt{\xi}}\left(\ln\left|\frac{-\alpha}{2\sqrt{\xi}-\alpha}\right| - \ln\left|\frac{\alpha}{2\sqrt{\xi}+\alpha}\right|\right)\right] - \frac{1}{2}\frac{1}{\sqrt{\xi}}\ln 1 + \frac{1}{2}\frac{1}{\sqrt{\xi}}\ln\left(\frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}\right) = \\
&= \lim_{\alpha\to 0}\frac{1}{2\sqrt{\xi}}\ln\left(\frac{2\sqrt{\xi}+\alpha}{2\sqrt{\xi}-\alpha}\right) + \frac{1}{2}\frac{1}{\sqrt{\xi}}\ln\left(\frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}\right) = \frac{1}{2}\frac{1}{\sqrt{\xi}}\ln\left(\frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}\right).
\end{aligned}
$$

For the imaginary part we obtain

$$
\int_0^1 dw \frac{1}{w+\sqrt{\xi}+i\eta} i\pi\delta(w-\sqrt{\xi}) = \frac{i\pi}{2\sqrt{\xi}}.
$$

The final result is then

$$
I_{\mu\nu} = -\frac{i\alpha}{3\pi}\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\ln\frac{M^2}{m^2} + i(q^2 g_{\mu\nu} - q_\mu q_\nu)\left[-\frac{5\alpha}{9\pi} - \frac{4\alpha}{3\pi}\frac{m^2}{q^2} - \frac{\alpha}{3\pi}\left(1+\frac{2m^2}{q^2}\right)f(q^2)\right],
$$

with a function $f(q^2)$ which is defined as

$$
f(q^2) = \begin{cases}
\sqrt{1-\frac{4m^2}{q^2}}\ln\frac{\sqrt{1-\frac{4m^2}{q^2}}+1}{\sqrt{1-\frac{4m^2}{q^2}}-1} & , \quad q^2 < 0, \\
2\sqrt{\frac{4m^2}{q^2}-1}\arctan\frac{1}{\sqrt{\frac{4m^2}{q^2}-1}} & , \quad 0 \le q^2 \le 4m^2, \\
\sqrt{1-\frac{4m^2}{q^2}}\ln\frac{\sqrt{1-\frac{4m^2}{q^2}}+1}{1-\sqrt{1-\frac{4m^2}{q^2}}} & , \quad q^2 > 4m^2.
\end{cases}
$$

8.2 Self-energy

The self-energy graph in Pauli-Villars regularisation is given by

$$
-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\left[\frac{-i}{k^2-\lambda^2+i\varepsilon}\gamma_\nu \frac{i}{\not{p}-\not{k}-m+i\varepsilon}\gamma^\nu - (\lambda\to M)\right].
$$

Here λ is an infinitesimal photon mass which will be taken to zero in the end. It is only introduced to make all expressions well defined. By using the identity

$$\frac{i}{k^2 - \lambda^2 + i\varepsilon} = \int_0^\infty dz_2 \exp\left(iz_2(k^2 - \lambda^2 + i\varepsilon)\right),$$

in combination with the Feynman parameter z_2 (plus doing the same with another parameter z_1) and substituting

$$k_\varrho = l_\varrho + p_\varrho \frac{z_1}{z_1 + z_2},$$

we obtain that

$$\begin{aligned}\Sigma &= ie^2 \int_0^\infty dz_2 \int_0^\infty dz_1 \int \frac{d^4 l}{(2\pi)^4} \left[\left(2\not{p}\frac{z_2}{z_1 + z_2} - 2\not{l} - 4m\right) \cdot \right. \\ &\quad \left. \cdot \exp(i(z_1 + z_2) l^2) \exp\left(ip^2 \frac{z_1 z_2}{z_1 + z_2} - iz_1 m^2 - iz_2 \lambda^2\right) - (\lambda \to M)\right].\end{aligned}$$

The l integration gives us contributions in form of

$$\begin{aligned}\int \frac{d^4 l}{(2\pi)^4} \exp\left(il^2 a\right) &= \frac{-i}{16\pi^2 a^2}, \\ \int \frac{d^4 l}{(2\pi)^4} l_\varrho \exp\left(il^2 a\right) &= 0.\end{aligned}$$

The next step is to rescale the integrale by introducing a new variable called γ - in combination with

$$1 = \int_0^\infty \frac{d\gamma}{\gamma} \delta\left(1 - \frac{z_1 + z_2}{\gamma}\right).$$

The rescaling is performed with $z_i = \gamma w_i$. We obtain

$$\begin{aligned}\Sigma &= \frac{\alpha}{4\pi} \int_0^1 dw_1 \int_0^1 dw_2 \left[2\not{p}\frac{w_2}{w_1 + w_2} - 4m\right] \frac{1}{(w_1 + w_2)^2} \delta(1 - w_1 - w_2) \cdot \\ &\quad \cdot \underbrace{\int_0^\infty \frac{d\gamma}{\gamma} \left[\exp(i\gamma a(\lambda)) - \exp(i\gamma a(M))\right]}_{= \log a(M)/a(\lambda)}, \\ a(x) &= p^2 \frac{w_1 w_2}{w_1 + w_2} - w_1 m^2 - w_2 x^2.\end{aligned}$$

Finally we can factor out the divergent part and see that

$$\Sigma(p)\big|_{\text{divergent}} = \frac{\alpha}{4\pi}\left[3m - (\not{p} - m)\right] \log \frac{M^2}{m^2}. \tag{8.1}$$

8.3 Vertex correction

The vertex correction graph is described by the following Feynman amplitude

$$\Lambda_\mu = -e^3 \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{k^2 - \lambda^2 + i\varepsilon}\frac{\gamma_\nu(\not{p}' - \not{k} + m)}{(p'-k)^2 - m^2 + i\varepsilon}\frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\nu}{(p-k)^2 - m^2 + i\varepsilon} - (\lambda \to M)\right].$$

We introduce an artifical variable z to reduce all integrals to a single one. We are able to obtain the factors k_ϱ by differentiating with respect to z^ϱ. After this differentiation z is put to zero. Since the divergent part is independent of p' and p we pick $p^2 = m^2$ and $p'^2 = m^2$ (we are on the mass shell). In case of the self-energy we could not look at this special case due to the p dependence (and later outcome of $m_R(p)$!). The basic integral looks like this

$$\begin{aligned} I \;&=\; \frac{1}{(4\pi)^2}\int_0^\infty dz_1 \int_0^\infty dz_2 \int_0^\infty dz_3 \frac{1}{(z_1+z_2+z_3)^2}\cdot \\ &\quad \dot{\exp}\left(-i\frac{\left(\frac{z_1}{2} - p'z_2 - pz_3\right)^2}{(z_1+z_2+z_3)^2} - i\lambda^2 z_1 - \varepsilon(z_1+z_2+z_3)\right). \end{aligned}$$

By rotating the integral in the negative imaginary axis we see that for $z_i \to -i\infty$ the integrals converge, because

$$\begin{aligned} z_1 \to -i\infty \quad &\Rightarrow \quad \exp(-\lambda^2\infty), \\ z_2 \to -i\infty \quad &\Rightarrow \quad \exp(-i(-\infty^2)/(-i\infty)) = \exp(-\infty), \\ z_3 \to -i\infty \quad &\Rightarrow \quad \exp(-\infty). \end{aligned}$$

To perform the Wick rotation we have to substitute $z_i = -i\alpha_i$. We need to replace

$$\int_0^\infty \cdots dz_i = -\int_{-i\infty}^0 \cdots dz_i.$$

We get something like

$$I = \frac{1}{(4\pi)^2}\int d\alpha_1 d\alpha_2 d\alpha_3 \frac{1}{(\alpha_1+\alpha_2+\alpha_3)^2}\exp\left[\frac{\left(\frac{\alpha_1}{2}+ip'\alpha_2+ip\alpha_3\right)^2}{(\alpha_1+\alpha_2+\alpha_3)^2}\right]\exp(-\lambda\alpha_1).$$

The (until now neglected) numerator can be recontructed by taking the appropriate derivates with respect to z and setting $z=0$ afterwards. By introducing the identity

$$1 = \int_0^\infty d\gamma\delta\left(\gamma-\alpha_1-\alpha_2-\alpha_3\right),$$

we have finally arrived at

$$\begin{aligned}\Lambda_\mu &= -\frac{e^3}{4\pi^2}\int_0^\infty \frac{d\gamma}{\gamma^3}\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2\int_0^\infty d\alpha_3\delta\left(\gamma-\alpha_1-\alpha_2-\alpha_3\right)\cdot \\ &\cdot\left\{\exp\left(-\frac{(p'\alpha_2+p\alpha_3)^2}{p}-\lambda^2\alpha_1\right)\left[\gamma_\mu\left(\gamma p'\cdot p-\frac{1}{2}(\alpha_2+\alpha_3)(p'+p)^2+\right.\right.\right. \\ &\left.\left.\left.+\frac{1}{2\gamma}(m^2(\alpha_2+\alpha_3)^2-\alpha_2\alpha_3q^2)-\frac{1}{2}\right)+\frac{m}{2}(p'_\mu+p_\mu)\frac{\alpha_1(\alpha_2+\alpha_3)}{\gamma}\right]-(\lambda\to M)\right\}.\end{aligned}$$

By subtituting $\alpha_i=\gamma\beta_i$ (rescaling) and then performing the γ integral (like in the self-energy section) we come to the integral

$$\int_0^\infty \frac{d\gamma}{\gamma}\left(\exp(ia\gamma)-\exp(ib\gamma)\right) = \log\frac{b}{a}.$$

Here we use that only the term with $1/2\gamma$ is divergent. The rest can be explained with the help of the Gordon-Identity,

$$\overline{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\overline{u}(p')\left[(p'+p)^\mu + i\sigma^{\mu\nu}(p'-p)_\nu\right]u(p). \tag{8.2}$$

We are only interested in the divergent part - which is then

$$\Sigma_\mu\big|_{\text{divergent}} = \gamma_\mu(-ie)\left(-\frac{\alpha}{4\pi}\log\frac{M^2}{m^2}\right). \tag{8.3}$$

The $\sigma^{\mu\nu}$ part is discussed in the next chapter - when we cross-check the QED with the $g-2$ experiment.

9 Magnetic moment of the electron and myon

9.1 The experiments to $g - 2$

We are looking at the additional term

$$\frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2),$$

with a function $F_2(q^2)$ that we calculated to be

$$F_2(q^2) = \frac{e^2}{\pi}\int_0^1 d\beta_1 \int_0^1 d\beta_2 \int_0^1 d\beta_3 \delta(1-\beta_1-\beta_2-\beta_3)\frac{m^2\beta_1(\beta_2+\beta_3)}{m^2(\beta_2+\beta_3)^2+\beta_1\lambda^2-\beta_2\beta_3 q^2}.$$

By looking at the limit for e.g. a weak magnetic field we get

$$\lim_{q^2\to 0}\lim_{\lambda^2\to 0} F_2(q^2) = \frac{\alpha}{2\pi}.$$

We could get this by using the Gordon-Identity, which is

$$\overline{u}(p')\gamma_\mu u(p) = \overline{u}(p')\left[\frac{(p+p')_\mu}{2m} + i\sigma_{\mu\nu}\frac{1}{2m}q^\nu\right]u(p). \tag{9.1}$$

The relation between the magnetic moment and this Gordon-Identity is given by the last term. For small momenta we can couple on more than one photon field, which leads to

$$-e\overline{u}(x)\frac{\sigma_{\mu\nu}}{2m}u(x)\partial^\nu A^\mu(x) = \frac{e}{4m}\overline{u}(x)\sigma_{\mu\nu}u(x)F^{\mu\nu}(x).$$

We used that $\sigma_{\mu\nu}$ is antisymmetric and gives us

$$\sigma_{\mu\nu}\partial^\nu A^\mu = \frac{1}{2}\left(\sigma_{\mu\nu}\partial^\nu A^\mu + \sigma_{\nu\mu}\partial^\mu A^\nu\right) = \frac{1}{2}\sigma_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\sigma_{\mu\nu}F^{\nu\mu}.$$

The magnetic field B^3 is given by e.g. $-F^{12}$. The coupling of F^{12} is therefore given by

$$\overline{u}\sigma_{12}u = 2\overline{u}\Sigma_3 u = 2\overline{u}\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1/2 \end{pmatrix} u.$$

So we get coupling of

$$-\frac{e}{4m}2\overline{u}\Sigma_3 u B^3 \frac{\alpha}{2\pi}.$$

For the overall coupling on the B^3 field we get an additional term, like shown in figure 9.1,

$$g = 2\left(1 + \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2)\right).$$

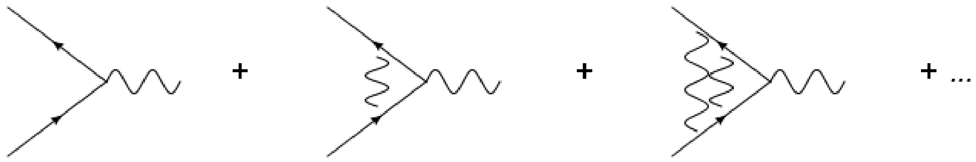

Figure 9.1: The contributions for the magnetic moment expressed in Feynman diagrams

There have been several experiments to proof this - with the most famous one mentioned in the Phys. Rev. Letter 92 (2004) 161802, with the results being

$$\begin{aligned} g^\mu_{\text{exp}} &= 2(1 + 0.0011659208(6)), \\ g^e_{\text{exp}} &= 2(1 + 0.0011596521859(38)), \\ g^e &= 2(1 + 0.0011596521539(240)), \\ \frac{\alpha}{2\pi} &= 0.001161. \end{aligned}$$

We will discuss now the most famous experiments to measure the required data.

9.2 The $(g-2)_\mu$ experiment

A magnetic dipol in a magnetic field has a torque (with $\vec{\mu}$ being the magnetic moment) of

$$\frac{d\vec{L}}{dt} = \vec{\mu} \times \vec{B}.$$

The main idea is to interpret the magnetic moment in a frequency. With the help of

$$\vec{\mu} = g\frac{e}{2mc}\vec{s}$$

we get the rotation frequency

$$\omega_s = g\frac{eB}{2mc}.$$

However a charged particle also moves on a so called Landau-Orbit, which gives us the Cyclotron frequency of

$$\omega_C = \frac{eB}{mc}.$$

Therefore for $g = 2$ we have $\omega_C = \omega_s$. The idea is now to place a detector in a certain angle to a magnetic field and detect the process

$$\mu^+ \to e^+ + \nu_e + \overline{\nu}_\mu.$$

The signal should oscillate in the same frequency. The big advantage is that a very large amount of data can be measured. To reduce some systematic errors several tricks like the usage of quadrupole magnets are required. The theoretical background is (from Jackson - Electrodynamics, Eq. (11.171))

$$\frac{d}{dt}(\vec{\beta}\vec{S}) = -\frac{e}{mc}s_\perp\left[\left(\frac{g}{2}-1\right)\vec{\beta}\times\vec{B} + \vec{E}\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\right].$$

In order to get the last term to zero we need

$$\left(\frac{g\beta}{2}-\frac{1}{\beta}\right) \stackrel{!}{=} 0 \quad \Rightarrow \quad \frac{g}{2} = \beta^{-2} \quad \Rightarrow \quad \gamma^2 = 1 + \frac{1}{a_\mu} \approx 29.3.$$

Here a_μ denotes the anomalous magnetic moment of the Myon. The main setup is shown in figure 9.2.

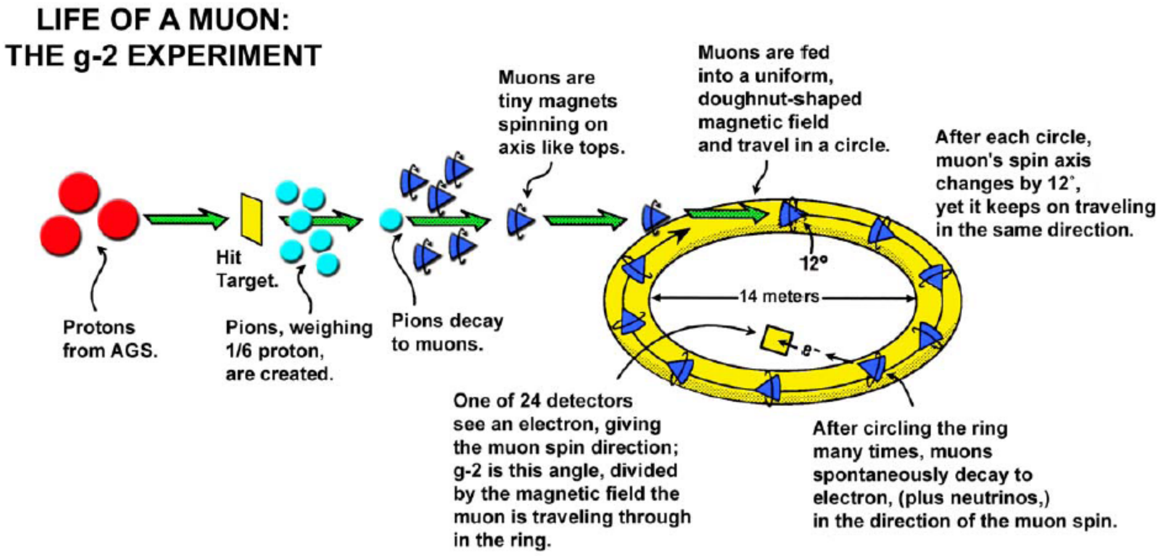

Figure 9.2: The setup for the $(g-2)_\mu$ experiment

9.3 The $(g-2)_e$ experiment

For this experiment a so called *Penning trap* was used to capture an electron. The concept is shown in Figure 9.3.

The potential is

$$\Phi(r,z) = \overbrace{\Phi_0}^{=0} + \Phi_{10}r^2 + \Phi_{01}z^2 + \underbrace{\Phi_{20}r^4 + \Phi_{02}z^4 + \Phi_{11}r^2z^2}_{=0}.$$

The last term is zero because of the choice of shells. This is exactly an harmonic oscillator in the z-direction with $\omega_Z \approx 60$ MHz. The solution of the Dirac equation is

$$\omega_Z \approx \omega_{z_0} - \frac{b\mu z}{m\omega}.$$

More exactly we find eigenstates with the energy

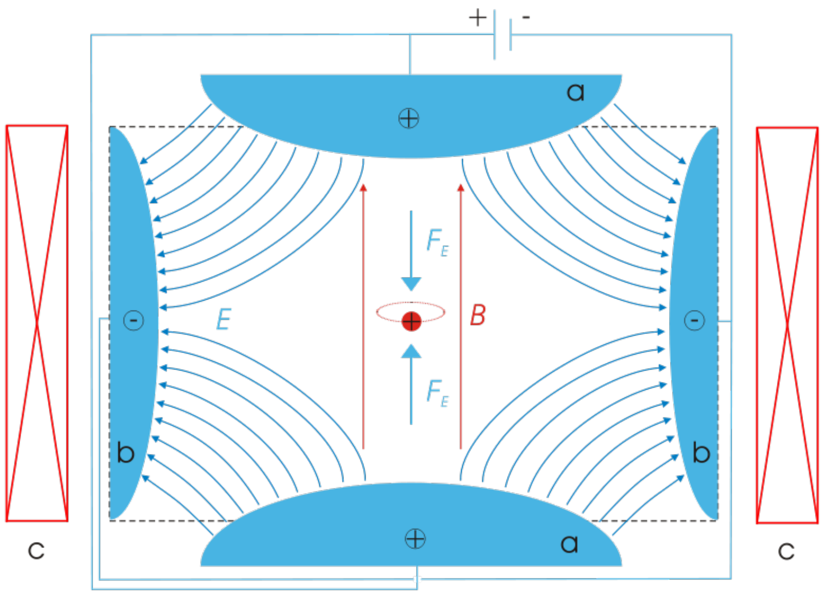

Figure 9.3: Drawing of a schematic Penning Trap (some kind of ion-cage) for the storage of charged particles by the use of a constant electric field (blue), generated by a quadrupole (a:end caps) and (b:ring electrode) and a superposed constant and homogeneous magnetic field (red), generated by a surrounding cylinder magnet (c). A particle, indicated in red (here positive) is stored in between caps of the same polarity. The particle is trapped inside a vacuum chamber.

$$E = \sqrt{m^2 + p_z^2 + (2n + 2s_z + 1)m\omega_c} + s_z \frac{a_e e B}{m}.$$

The task is then to measure how often the trapped electron changes it's spin. As soon as the electron is in the ground state $n = 0$ it will be pumped up to $n = 1$ again.

10 Euler-Heisenberg Lagrange density

The vacuum is an optically active media as shown in figure 10.1.

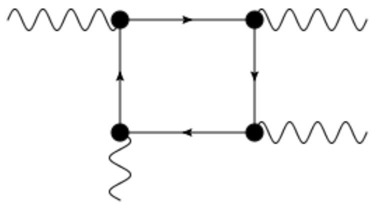

Figure 10.1: Vacuum fluctuations in quantum electrodynamics due to photons

The leads to terms which are proportional to $(\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E} \cdot \vec{B})^2$. This is important, because

- it has an application in describing magnetars or massive ion scattering.
- it is an example for an effective field theory.

An elementary theory at high energy scales at low momenta can be described as an effective field theory. Example are:

- Proton vs. Quarks + Gluons,
- The process shown below, where we can exchange the unknown process by just one vertex.

This is as we know not allowed, but we argue that we interaction is weak. If it would be stronger or strong enough we would be able to see the interaction in detail and thus would not have just a single vertex but the whole interaction picture.

- This is also quantum electrodynamics with complete other techniques (so called Schwinger formalism). The problem is that it is quite hard to calculate processes using this formalism.

The Schwinger formalism transforms the problem of quantum electrodynamics in static $\vec{E}$ and $\vec{B}$ fields into a quantum mechanical problem. We have

$$\mathcal{L} = \mathcal{L}(A_\mu^{(0)}) + \mathcal{L}^{(1)}(\delta A_\mu).$$

We say that $A_\mu^{(0)}$ is the background potential, while the δA_μ is for example a propagating photon. The gauge invariant momentum operator is then given by

$$\hat{P}_\mu = \hat{p}_\mu - eA_\mu^{(0)}(\hat{X}),$$

where $\hat{X}$ is the position operator. We introduce position eigenfunctions, called $\|x\rangle\rangle$. We have

$$\begin{aligned} \hat{P}_\mu\|x\rangle\rangle &= i\partial_\mu\|x\rangle\rangle, \\ \hat{X}\|x\rangle\rangle &= x\|x\rangle\rangle. \end{aligned}$$

The $\mathcal{L}^{(1)}$ term can be calculated. We see that

$$\begin{aligned} \mathcal{L}^{(1)} &= -e\delta\hat{A}^\mu(\hat{X})\hat{\overline{\psi}}\gamma_\mu\hat{\psi}(\hat{X}) = e\delta\hat{A}^\mu(\hat{X})\mathrm{tr}\left\{\gamma_\mu\hat{\psi}(\hat{X})\hat{\overline{\psi}}(\hat{X})\right\} = \\ &= ie\mathrm{tr}\left\{\delta\hat{A}^\mu(\hat{X})\gamma_\mu\frac{\hat{\not{p}}+m}{p^2-m^2+i\varepsilon}\right\} = \\ &= -\mathrm{tr}\left\{(\delta\hat{\not{P}})\hat{\not{P}}\int_0^\infty ds\exp\left(-is\left[\hat{\not{p}}^2-m^2+i\varepsilon\right]\right)\right\} = \\ &= -\frac{1}{2}\delta\left[\mathrm{tr}\left\{\int_0^\infty\frac{ds}{is}\exp\left(is\left[\hat{\not{p}}^2-m^2+i\varepsilon\right]\right)\right\}\right]. \end{aligned}$$

Therefore we have

$$\int d^4x\delta\mathcal{L}(x) = \frac{i}{2}\delta\left[\int_0^\infty \frac{ds}{s}\exp(-im^2s)\mathrm{tr}\left\{\langle\langle x\|\exp(-is\underbrace{(\hat{\not{p}}^2+i\varepsilon)}_{=\mathcal{H}})\|x\rangle\rangle\right\}\right].$$

We say that s is Schwinger's *proper time* variable. The calculation is now based on the way to handle this from quantum mechanics. We see that

$$\begin{aligned}
\frac{d\hat{X}_\mu}{ds} &= -i\left[\hat{X}_\mu,\mathcal{H}\right] = -2\hat{P}_\mu,\\
\frac{d\hat{P}_\mu}{ds} &= -i\left[\hat{P}_\mu,\mathcal{H}\right] = -i\left(\left[\hat{P}_\mu,\hat{P}_\nu\right]\hat{P}^\nu + \hat{P}_\nu\left[\hat{P}_\mu,\hat{P}^\nu\right]\right) =\\
&= -2eF^{(0)}_{\mu\nu}(\hat{X})\hat{P}^\nu.
\end{aligned}$$

We set $\mathcal{F} = (F_{\mu\nu})$ for the whole matrix. The solution of the differential equation is then

$$(\hat{P}^\mu)(s) = \exp(-2e\mathcal{F}s)(\hat{P}^\mu)(0). \tag{10.1}$$

So we have for $\hat{X}$:

$$\frac{d(\hat{X}^\mu)(s)}{ds} = -2(\hat{P}^\mu)(s). \tag{10.2}$$

By adding equation (10.1) to equation (10.2) we obtain

$$(\hat{P}^\mu)(s) = e\mathcal{F}\exp(-2\mathcal{F}s)\left(\exp(-2e\mathcal{F}s)-1\right)^{-1}\left((\hat{X}^\mu)(s)-(\hat{X}^\mu)(0)\right).$$

After all the efforts we finally have found

$$\begin{aligned}
\hat{P}_\mu\hat{P}^\mu &= -\frac{i}{2}\mathrm{tr}\left\{e\mathcal{F}\mathrm{cotanh}(e\mathcal{F}s)\right\},\\
\hat{\not{P}}\hat{\not{P}} &= \hat{P}_\mu\cdot\hat{P}^\mu - \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}.
\end{aligned}$$

Therefore we see that our additional Lagrange term is now

$$\begin{aligned}
\mathcal{L}^{(1)}(x) &= \frac{1}{32\pi^2}\int_0^\infty ds\frac{1}{s^3}\exp(-ims^2-\varepsilon s)\exp\left(\frac{1}{2}\mathrm{tr}\left\{\ln\frac{\sinh(e\mathcal{F}s)}{e\mathcal{F}s}\right\}\right)\cdot\\
&\quad\cdot\mathrm{tr}\left\{\exp\left(\frac{i}{2}e\sigma_{\mu\nu}\mathcal{F}s\right)\right\} =\\
&= -\frac{1}{8\pi^2}\int_0^\infty dss^{-3}\underbrace{\exp(-m^2s)\vec{E}\cdot\vec{B}(es)^2\frac{\mathrm{Re}(\cosh(es\sqrt{-\vec{E}^2+\vec{B}^2+2i\vec{E}\vec{B}})}{\mathrm{Im}(es\sqrt{-\vec{E}^2+\vec{B}^2+2i\vec{E}\vec{B}})}}_{\equiv C}.
\end{aligned}$$

This is an exact result for constant $\vec{E}$ and $\vec{B}$ in all orders, which is quite remarkable. We will now do an approximation (Taylor expansion) for $A, q \ll m$. Therefore we have $\vec{E}, \vec{B} \ll m^2$. We know that

$$\cosh\left(es\sqrt{\vec{B}^2 - \vec{E}^2 + 2i\vec{E}\cdot\vec{B}}\right) = 1 + \frac{1}{2}(es)^2\left(\vec{B}^2 - \vec{E}^2 + 2i\vec{E}\cdot\vec{B}\right)^2 + \mathcal{O}(4).$$

If we do this to the 4th order we see that

$$C = 1 + \frac{1}{3}(es)^2 \underbrace{\left(\vec{B}^2 - \vec{E}^2\right)}_{=\frac{1}{2}F_{\mu\nu}F^{\mu\nu}} + (es)^4 \left[-\frac{1}{45}\left(\vec{B}^2 - \vec{E}^2\right)^2 - \frac{2}{45}\left(\vec{E}\vec{B}\right)^2\right].$$

Since the first term is only a constant we can neglect it. The second term gives us the well known wavefunction renormalization (coming from A^μ). The third term is convergent and gives us

$$\mathcal{L}_{\text{eff}}^R = \frac{2\alpha^2}{45}\frac{1}{m^4}\left[\left(\vec{B}^2 - \vec{E}^2\right) + 7\left(\vec{E}\cdot\vec{B}\right)^2\right]. \tag{10.3}$$

This is the so called Euler-Heisenberg Lagrangean. Some magnetic field values are:

- The earth's magnetic field is around 0.6 G.
- The strongest laboratory magnet gives us 10^6 G.
- The magnetic field of a pulsar is 10^{13} G. At such strong fields the atoms are not *round* any more, but look more like pins. This gave astro physicists a hard spectrum to analyze - but after this was understood they were able to calculate the magnetic field strength by using the data.
- The magnetic field of a magnetar is 10^{15} G.
- Very massive ion collisions have like 10^{17} G and more at the LHC. Such strong field correspond to an energy of $(100 \text{ MeV})^2$.